Differential Geometry

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Abstract

This note is based on Differential Geometry, taught by Xiang Ma in Fall 2025 at Peking University, and also refers to Differential Geometry by Weihuan Chen.

Contents

1 Curves

1.1 Curvature and Torsion

Definition 1.1. A parametrized curve $\boldsymbol{r}(t)$ is **regular** if $\boldsymbol{r}'(t) \neq 0$ for all t in its domain.

Definition 1.2. Given a regular curve $\boldsymbol{r}(t)$, its **arc length** from $t = a$ to $t = b$ is defined as

$$s = \int_a^b |\boldsymbol{r}'(t)| \mathrm{d}t. \tag{1}$$

We can reparametrize the curve by its arc length s such that

$$\mathrm{d}s = |\boldsymbol{r}'(t)| \mathrm{d}t. \tag{2}$$

Such a parameter is called the **arc length parameter**.

Let $\boldsymbol{T} : \boldsymbol{r} \to S^1$ be the unit tangent vector field along the curve $\boldsymbol{r}(s)$, i.e. $\boldsymbol{T}(s) = \boldsymbol{r}'(s)$. Assume θ is the angle between $\boldsymbol{T}(s)$ and the x-axis.

Definition 1.3. The **curvature** of the curve $\boldsymbol{r}(s)$ at s is defined as

$$\kappa(s) = |\boldsymbol{T}'(s)| = \left|\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right| = |\boldsymbol{r}''(s)|. \tag{3}$$

Let $\boldsymbol{T} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Then $\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s} = \theta' \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. Define the unit normal vector $\boldsymbol{N} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. Then we have

$$\boldsymbol{T}' = \kappa \boldsymbol{N}. \tag{4}$$

Because $\boldsymbol{T} \cdot \boldsymbol{T} = 1$, we have $\boldsymbol{N} \cdot \boldsymbol{T} = \boldsymbol{T}' \cdot \boldsymbol{T} = 0$. Thus $\boldsymbol{N}$ is orthogonal to $\boldsymbol{T}$. In the two dimensional space, $\{\boldsymbol{r}(s); \boldsymbol{T}(s), \boldsymbol{N}(s)\}$ forms an right-hand oriented orthonormal frame along the curve $\boldsymbol{r}(s)$, called the **Frenet frame**. And we have the **Frenet formula**:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix}. \tag{5}$$

Example 1.1. Consider the ellipse $\boldsymbol{r}(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}$. We have $\boldsymbol{r}'(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$ and $|\boldsymbol{r}'(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$. Thus the arc length parameter is $s = \int_0^t |\boldsymbol{r}(t)| \,\mathrm{d}t$, which is an elliptic integral.

To avoid the elliptic integral in Example 1.1, we can compute the Frenet frame and curvature without reparametrization. Since $\boldsymbol{r}'(t) = |\boldsymbol{r}'(t)|\, \boldsymbol{T}(t)$, we have

$$\boldsymbol{r}''(t) = |\boldsymbol{r}'(t)| \frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\mathrm{d}\,|\boldsymbol{r}'(t)|}{\mathrm{d}t}\boldsymbol{T}(t) = |\boldsymbol{r}'(t)|^2 \kappa(t)\boldsymbol{N}(t) + \frac{\mathrm{d}\,|\boldsymbol{r}'(t)|}{\mathrm{d}t}\boldsymbol{T}(t),$$

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = |\boldsymbol{r}'(t)|^3 \kappa(t)\boldsymbol{T}(t) \times \boldsymbol{N}(t) = |\boldsymbol{r}'(t)|^3 \kappa(t)\boldsymbol{B}(t).$$

Thus we have

$$\kappa(t) = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}, \quad \boldsymbol{B}(t) = \frac{\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)}{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}, \quad \boldsymbol{N}(t) = \boldsymbol{B}(t) \times \boldsymbol{T}(t). \tag{6}$$

And $\{\boldsymbol{r}(s); \boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ forms the Frenet frame along the curve $\boldsymbol{r}(t)$.

Example 1.2. Consider the ellipse in Example 1.1. We have

$$\boldsymbol{r}''(t) = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}, \quad \kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

Definition 1.4. The **center of curvature** of the curve $\boldsymbol{r}(s)$ at s is defined as

$$\tilde{\boldsymbol{r}}(s) = \boldsymbol{r}(s) + \frac{1}{\kappa(s)}\boldsymbol{N}(s). \tag{7}$$

The image of the curve $\tilde{\boldsymbol{r}}(s)$ is called the **evolute** of the curve $\boldsymbol{r}(s)$. The **osculating circle** of the curve $\boldsymbol{r}(s)$ at s is the circle with center $\tilde{\boldsymbol{r}}(s)$ and radius $\frac{1}{\kappa(s)}$.

Notice the fact that for s such that $\kappa(s)$ attains its extrema points, $\tilde{\boldsymbol{r}}(s)$ often has a cusp. Since

$$\tilde{\boldsymbol{r}}'(s) = \boldsymbol{T}(s) + \frac{1}{\kappa(s)}\left(-\kappa(s)\right)\boldsymbol{T}(s) + \left(\frac{-\kappa'(s)}{\kappa(s)^2}\right)\boldsymbol{N}(s) = -\frac{\kappa'(s)}{\kappa(s)^2}\boldsymbol{N}(s),$$

we have $\tilde{\boldsymbol{r}}'(s) = 0$ when $\kappa'(s) = 0$.

Definition 1.5. $\boldsymbol{N}(s)$ is the **principal normal vector**, $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$ is the **binormal vector**. $\operatorname{span}\{\boldsymbol{T}(s), \boldsymbol{N}(s)\}$ is the **osculating plane** of the curve $\boldsymbol{r}(s)$ at s.

Definition 1.6. Let $\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}s} = -\tau\boldsymbol{N}$. Then τ is called the **torsion** of the curve $\boldsymbol{r}(s)$ at s.

Thus we have the Frenet formula in $\mathbb{R}^3$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}. \tag{8}$$

By the Frenet formula, we have

$$\tau = \frac{(\boldsymbol{r}' \times \boldsymbol{r}'') \cdot \boldsymbol{r}'''}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2}. \tag{9}$$

Corollary 1.1. *κ, τ do not depend on t and they are invariant under rigid motions.*

1.2 The Local Canonical Form

Let $\boldsymbol{r}(s)$ be a regular curve parametrized by arc length. Consider expanding $\boldsymbol{r}(s)$ at $s = 0$ by Taylor expansion:

$$\begin{aligned} \boldsymbol{r}(s) &= \boldsymbol{r}(0) + s\boldsymbol{r}'(0) + \frac{s^2}{2!}\boldsymbol{r}''(0) + \frac{s^3}{3!}\boldsymbol{r}'''(0) + o(s^3) \\ &= \boldsymbol{r}(0) + \left(s - \frac{\kappa(0)^2}{6}s^3\right)\boldsymbol{T}(0) + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right)\boldsymbol{N}(0) + \frac{\kappa(0)\tau(0)}{6}s^3\boldsymbol{B}(0) \\ &\quad + o(s^3). \end{aligned} \tag{10}$$

Definition 1.7. The **local canonical form** of the curve $\boldsymbol{r}(s)$ at $s = 0$ is given by

$$x(s) = s - \frac{\kappa(0)^2}{6}s^3, \quad y(s) = \frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3, \quad z(s) = \frac{\kappa(0)\tau(0)}{6}s^3. \tag{11}$$

Definition 1.8. Ignoring the higher order term, the **approximate curve** of $\boldsymbol{r}(s)$ at $s = 0$ is given by $\tilde{\boldsymbol{r}}(s) = \left(s, \frac{\kappa(0)}{2}s^2, \frac{\kappa(0)\tau(0)}{6}s^3\right)$.

1.3 Fundamental Theorem of Curves

Theorem 1.1 (Fundamental Theorem of Curves)**.** *Given* $s \in [a,b]$ *and* $\kappa(s) > 0, \tau(s)$ *are continuous, there exists a regular curve* $\boldsymbol{r}(s) : [a,b] \to \mathbb{R}^3$ *parametrized by arc length such that* $\kappa(s), \tau(s)$ *are its curvature and torsion. Moreover, any other curve satisfying the same conditions differs from* $\boldsymbol{r}(s)$ *by a rigid motion. If* $\boldsymbol{r}_1$ *and* $\boldsymbol{r}_2$ *are two curves satisfying the same conditions, then there exists* $\sigma \in SO(3)$ *and* $\boldsymbol{r}_0 \in \mathbb{R}^3$ *such that* $\boldsymbol{r}_2(s) = F(\boldsymbol{r}_1(s))$ *for all* $s \in [a,b]$*.*

2 Surfaces

2.1 Gauss Equation and Weingarten Equation

For a surface $S \subset \mathbb{R}^3$, taking $\{\boldsymbol{r}(u,v); \boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}\}$ as a local parametrization of S, our goal is to find the relations between $\boldsymbol{r}_{uu}, \boldsymbol{r}_{uv}, \boldsymbol{r}_{vv}, \boldsymbol{n}_u, \boldsymbol{n}_v$ and $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{n}$. Recall that we have already known the **Weingarten equation**:

$$\begin{cases} \boldsymbol{n}_u = -a_{11}\boldsymbol{r}_u - a_{12}\boldsymbol{r}_v, \\ \boldsymbol{n}_v = -a_{21}\boldsymbol{r}_u - a_{22}\boldsymbol{r}_v. \end{cases} \tag{12}$$

We denote **Gauss equation** as

$$\begin{cases} \boldsymbol{r}_{uu} = \Gamma_{11}^1\boldsymbol{r}_u + \Gamma_{11}^2\boldsymbol{r}_v + L\boldsymbol{n}, \\ \boldsymbol{r}_{uv} = \Gamma_{12}^1\boldsymbol{r}_u + \Gamma_{12}^2\boldsymbol{r}_v + M\boldsymbol{n}, \\ \boldsymbol{r}_{vv} = \Gamma_{22}^1\boldsymbol{r}_u + \Gamma_{22}^2\boldsymbol{r}_v + N\boldsymbol{n}, \end{cases} \tag{13}$$

where Γ_{ij}^k are the **Christoffel symbols**.

Doing inner products on both sides of the first equation with $\boldsymbol{r}_u, \boldsymbol{r}_v$, we have

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_u = \frac{1}{2}\left(\boldsymbol{r}_u\cdot\boldsymbol{r}_u\right)_u = \frac{E_u}{2}, \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_v = F_u - \frac{1}{2}E_v. \end{cases}$$

$$\Longleftrightarrow \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \frac{E_u}{2} \\ F_u - \frac{E_v}{2} \end{pmatrix}.$$

We take orthogonal parameters such that $F = 0$. Then we have

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{E_u}{2E} \\ -\frac{E_v}{2G} \end{pmatrix}, \quad \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{E_v}{2E} \\ \frac{G_u}{2G} \end{pmatrix}, \quad \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} -\frac{G_u}{2E} \\ \frac{G_v}{2G} \end{pmatrix}.$$

Partial differentiating the first two Gauss equations with v and u respectively, we have

$$\begin{aligned} \left(\boldsymbol{r}_{uu}\right)_v = {} & \left(\Gamma_{11}^1\right)_v \boldsymbol{r}_u + \left(\Gamma_{11}^2\right)_v \boldsymbol{r}_v + L_v\boldsymbol{n} + \Gamma_{11}^1\left(\Gamma_{12}^1\boldsymbol{r}_u + \Gamma_{12}^2\boldsymbol{r}_v + M\boldsymbol{n}\right) \\ & + \Gamma_{11}^2\left(\Gamma_{22}^1\boldsymbol{r}_u + \Gamma_{22}^2\boldsymbol{r}_v + N\boldsymbol{n}\right) + L\left(-a_{21}\boldsymbol{r}_u - a_{22}\boldsymbol{r}_v\right) \end{aligned}$$

$$\begin{aligned} \left(\boldsymbol{r}_{uv}\right)_u = {} & \left(\Gamma_{12}^1\right)_u \boldsymbol{r}_u + \left(\Gamma_{12}^2\right)_u \boldsymbol{r}_v + M_u\boldsymbol{n} + \Gamma_{12}^1\left(\Gamma_{11}^1\boldsymbol{r}_u + \Gamma_{11}^2\boldsymbol{r}_v + L\boldsymbol{n}\right) \\ & + \Gamma_{12}^2\left(\Gamma_{12}^1\boldsymbol{r}_u + \Gamma_{12}^2\boldsymbol{r}_v + M\boldsymbol{n}\right) + M\left(-a_{11}\boldsymbol{r}_u - a_{12}\boldsymbol{r}_v\right). \end{aligned}$$

Comparing the coefficients of $\boldsymbol{r}_u$, we have $0 = 0$. Comparing $\boldsymbol{r}_v$, we have **Gauss Equation**:

$$K = \frac{-1}{\sqrt{EG}}\left(\left(\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\right)_v + \left(\frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\right)_u\right). \tag{14}$$

Taking the curvature-line parameters, we have $F = 0 = M$ and **Codazzi Equations**:

$$L_v = HE_v, \quad N_u = HG_u. \tag{15}$$

2.2 Complex Coordinates on Surfaces

For a surface (M^2, I), take a (local) orthogonal parametrization such that

$$I = E\mathrm{d}u^2 + G\mathrm{d}v^2 = \left(\sqrt{E}\mathrm{d}u + \sqrt{G}\mathrm{d}vi\right)\left(\sqrt{E}\mathrm{d}u - \sqrt{G}\mathrm{d}vi\right).$$

Definition 2.1. If I can be written as $I = e^{2\rho}\left(\mathrm{d}u^2 + \mathrm{d}v^2\right)$ for some function ρ, then (u, v) is called an **isothermal parameter**.

Theorem 2.1 (Gauss)**.** *For (M, I), there always exists (local) isothermal parameters.*

Lemma 2.1. *For holomorphic order-one differential forms $\omega = \sqrt{E}\mathrm{d}u + \sqrt{G}\mathrm{d}vi$ on the surface, there exists a integrating factor ρ and complex function $z = u + iv$ such that*

$$\mathrm{d}z = e^{-\rho}\omega \Longleftrightarrow \sqrt{E}\mathrm{d}u + \sqrt{G}\mathrm{d}vi = e^{\rho}\mathrm{d}z.$$

From now on, we always take isothermal parameters (u, v) and complex coordinate $z = u + iv$. Thus we have

$$I = e^{2\rho}\left(\mathrm{d}u^2 + \mathrm{d}v^2\right) = e^{2\rho}\left|\mathrm{d}z\right|^2. \tag{16}$$

Definition 2.2. Recall the complex partial derivatives:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right), \quad \frac{\partial^2}{\partial z\partial \bar{z}} = \frac{\partial^2}{\partial \bar{z}\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

Recall the following facts:

- $f = f(z)$ is holomorphic $\Longleftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$;
- $\mathrm{d}f = f_u\mathrm{d}u + f_v\mathrm{d}v = f_z\mathrm{d}z + f_{\bar{z}}\mathrm{d}\bar{z}$;

Taking $\{\boldsymbol{r}; \boldsymbol{r}_z, \boldsymbol{r}_{\bar{z}}, \boldsymbol{n}\}$, which are complex vectors, we have

$$e^{2\rho} = \boldsymbol{r}_u \cdot \boldsymbol{r}_u = \boldsymbol{r}_v \cdot \boldsymbol{r}_v, \quad 0 = \boldsymbol{r}_u \cdot \boldsymbol{r}_v \Longleftrightarrow \boldsymbol{r}_z \cdot \boldsymbol{r}_z = 0, \quad \boldsymbol{r}_z \cdot \boldsymbol{r}_{\bar{z}} = \frac{e^{2\rho}}{2}.$$

Actually, $\boldsymbol{r}_z \cdot \boldsymbol{r}_z = 0$ if and only if (u, v) are isothermal parameters.

Proposition 2.1. *The moving equations of $\{\boldsymbol{r}; \boldsymbol{r}_z, \boldsymbol{r}_{\bar{z}}, \boldsymbol{n}\}$ are given by*

$$\begin{cases} \boldsymbol{r}_{zz} = 2\rho_z\boldsymbol{r}_z + Q\boldsymbol{n}, \\ \boldsymbol{r}_{z\bar{z}} = \frac{e^{2\rho}}{2}H\boldsymbol{n}, \\ \boldsymbol{n}_z = -H\boldsymbol{r}_z - 2e^{-2\rho}Q\boldsymbol{r}_{\bar{z}}, \end{cases} \tag{17}$$

where $Q = \boldsymbol{r}_{zz} \cdot \boldsymbol{n} = \frac{1}{4}\left(\boldsymbol{r}_{uu} - \boldsymbol{r}_{vv} - 2i\boldsymbol{r}_{uv}\right) \cdot \boldsymbol{n} = \frac{1}{4}(L - N - 2iM)$.

Proof. The first equation is from

$$\boldsymbol{r}_{zz} \cdot \boldsymbol{r}_z = \frac{1}{2}\left(\boldsymbol{r}_z \cdot \boldsymbol{r}_z\right)_z = 0, \quad \boldsymbol{r}_{zz} \cdot \boldsymbol{r}_{\bar{z}} = \left(\boldsymbol{r}_z \cdot \boldsymbol{r}_{\bar{z}}\right)_z - \boldsymbol{r}_z \cdot \boldsymbol{r}_{z\bar{z}} = e^{2\rho}\rho_z.$$

The second equation is from

$$\boldsymbol{r}_{z\bar{z}} \cdot \boldsymbol{n} = \frac{1}{4}\left(\boldsymbol{r}_{uu} + \boldsymbol{r}_{vv}\right) \cdot \boldsymbol{n} = \frac{1}{4}(L + N) = \frac{e^{2\rho}}{2}H.$$

The third equation is from

$$\frac{\boldsymbol{n}_z \cdot \boldsymbol{r}_{\bar{z}}}{\boldsymbol{r}_z \cdot \boldsymbol{r}_{\bar{z}}} = \frac{-\boldsymbol{n} \cdot \boldsymbol{r}_{z\bar{z}}}{\frac{1}{2}e^{2\rho}} = -H, \quad \frac{\boldsymbol{n}_z \cdot \boldsymbol{r}_z}{\boldsymbol{r}_z \cdot \boldsymbol{r}_{\bar{z}}} = \frac{-\boldsymbol{n} \cdot \boldsymbol{r}_{zz}}{\frac{1}{2}e^{2\rho}} = -2e^{-2\rho}Q.$$

□

Definition 2.3. $Q\mathrm{d}z^2$ is called the **Hopf differential** of the surface, which is a invariant differential form.

We can verify that

$$Q\mathrm{d}z^2 = -\left(\boldsymbol{r}_z \cdot \boldsymbol{n}_z\right)\mathrm{d}z^2 = -\left(\boldsymbol{r}_z \mathrm{d}z\right)\left(\boldsymbol{n}_z \mathrm{d}z\right).$$

In complex analysis, we know that by substituting $w = w(z)$, which is holomorphic, we have $\boldsymbol{r}_z\mathrm{d}z = \boldsymbol{r}_w\mathrm{d}w$.

Now consider the compatibility conditions: $(\boldsymbol{r}_{zz})_{\bar z} = (\boldsymbol{r}_{z\bar z})_z$ and $(\boldsymbol{n}_z)_{\bar z} = (\boldsymbol{n}_{\bar z})_z$.

$$(\boldsymbol{r}_{zz})_{\bar z} = \left(2\rho_z \boldsymbol{r}_z + Q\boldsymbol{n}\right)_{\bar z} = 2\rho_{z\bar z}\boldsymbol{r}_z + Q_{\bar z}\boldsymbol{n} + 2\rho_z\left(\frac{H}{2}e^{2\rho}\boldsymbol{n}\right) + Q\boldsymbol{n}_{\bar z},$$

$$(\boldsymbol{r}_{z\bar z})_z = \left(\frac{H}{2}e^{2\rho}\boldsymbol{n}\right)_z = \left(\frac{H_z}{2} + H\rho_z\right)e^{2\rho}\boldsymbol{n} + \frac{H}{2}e^{2\rho}\left(-H\boldsymbol{r}_z - 2e^{-2\rho}Q\boldsymbol{r}_{\bar z}\right).$$

Comparing the coefficients of $\boldsymbol{r}_z, \boldsymbol{n}$, we have

$$2\rho_{z\bar z} - 2e^{-2\rho}Q\overline{Q} = -\frac{H^2}{2}e^{2\rho}, \quad Q_{\bar z} = \frac{e^{2\rho}}{2}H_z. \tag{18}$$

Remark: The first equation is equivalent to the Gauss equation. The second equation shows that if the surface is a constant mean curvature surface, then $Q\mathrm{d}z^2$ is a holomorphic order-two differential form.

Theorem 2.2 (Hopf)**.** *M^2 is a regular surface with constant mean curvature in $\mathbb{R}^3$. If $M^2 \cong S^2$, then M^2 is a sphere.*

Proof. We have the following fact: a holomorphic order-two differential form on $S^2 = \mathbb{C} \cup \{\infty\}$ must be 0. Thus

$$Q \equiv 0 = L - N - 2iM \Longrightarrow L = N, M = 0 \Longrightarrow k_1 = \frac{L}{E} = \frac{N}{G} = k_2.$$

Hence M^2 is totally umbilic. We know that the only totally umbilic surfaces in $\mathbb{R}^3$ are planes and spheres. Therefore, M^2 is a sphere. □

Now we introduce the **Weierstrass representation** of a minimal surface. $H \equiv 0 \Longrightarrow \boldsymbol{r}_{z\bar z} = 0 \Longrightarrow \boldsymbol{r}_z = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ is holomorphic and since $|\boldsymbol{r}_z| = 0$, we have $(f_1)^2 + (f_2)^2 + (f_3)^2 = 0$.

Proposition 2.2. *The Weierstrass representation of a minimal surface is given by*

$$\boldsymbol{r}_z = \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(G - \frac{1}{G}\right) \\ -\frac{i}{2}\left(G + \frac{1}{G}\right) \\ 1 \end{pmatrix} h'(z), \quad \boldsymbol{r} = 2\Re\int_{z_0}^{z} \begin{pmatrix} \frac{1}{2}\left(G - \frac{1}{G}\right) \\ -\frac{i}{2}\left(G + \frac{1}{G}\right) \\ 1 \end{pmatrix} h'(z)\mathrm{d}z, \tag{19}$$

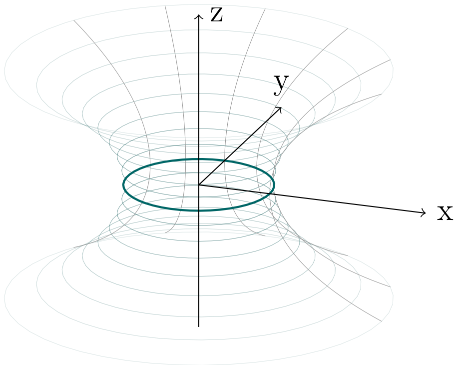
where $G(z)$ and $h(z)$ are holomorphic functions on some region.

Remark: $h'(z)\mathrm{d}z = \mathrm{d}h(z)$ is called **height differential**, which is a holomorphic differential form. $G = \frac{f_1 + if_2}{f_3}$ is a meromorphic function on Ω and $G\mathrm{d}h, \frac{1}{G}\mathrm{d}h$ have no poles. G corresponds to the Gauss map of the surface.

When Ω is simply connected, the integral is path-independent. Otherwise, $\boldsymbol{r}$ is well-defined if and only if $\oint_\gamma \boldsymbol{r}_z\mathrm{d}z$ does not has a real period for any closed curve γ in Ω.

Example 2.1. $\Omega = \mathbb{C}\backslash\{0\}, \quad G(z) = z, \quad \mathrm{d}h = h'(z)\mathrm{d}z = \frac{\mathrm{d}z}{z}.$

$$\boldsymbol{r} = 2\Re \int_{z_0}^{z} \begin{pmatrix} \frac{1}{2}\left(z - \frac{1}{z}\right) \\ -\frac{i}{2}\left(z + \frac{1}{z}\right) \\ 1 \end{pmatrix} \frac{\mathrm{d}z}{z} = 2\Re \begin{pmatrix} \frac{1}{2}\left(z + \frac{1}{z}\right) \\ -\frac{i}{2}\left(z - \frac{1}{z}\right) \\ \ln z \end{pmatrix} = 2 \begin{pmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{pmatrix}, \quad z = e^{u+iv}.$$



Remark: If $\mathrm{d}h = \frac{i\mathrm{d}z}{z}$, then we get the helicoid.

3 2D Manifolds and Global Geometry

3.1 Regular Surfaces

Definition 3.1. A subset $S \subset \mathbb{R}^3$ is a **regular surface** if

1. For each point $p \in S$, there exists an open neighborhood $V_p \subset \mathbb{R}^3$ of p and a parameter domain $U_p \subset \mathbb{R}^2$ such that $\boldsymbol{r} : U_p \to V_p \cap S$ is a bijection and continuous with continuous inverse;

2. $\boldsymbol{r}(u, v)$ is a regular parametrized surface.

Example 3.1.

$$0 = f(x, y, z) = x^2 + y^2 + z^2 - r^2.$$

Example 3.2.

$$0 = g(x, y, z) = x^2 + y^2 - z^2 - a.$$

Proposition 3.1 (Monge Patch)**.** *$S = \{(x, y, f(x, y))|(x, y) \in U \subset \mathbb{R}^2, f \in C^3\}$ is a regular surface.*

Proposition 3.2 (Regular Value Preimage Theorem)**.** *$F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable and for amy $\lambda \in \mathbb{R}$ and $p \in F^{-1}(\lambda)$, we have $grad_p(F) \neq \mathbf{0}$. (We call λ a **regular value** of F.) Then $S = F^{-1}(\lambda)$ is a regular surface in $\mathbb{R}^3$.*

Proof. For any $p \in F^{-1}(\lambda)$, WLOG, we assume that $F_z(p) \neq 0$. By the implicit function theorem, there exists a neighborhood V_p of p and a function $z = f(x, y)$ such that $F(x, y, f(x, y)) \equiv \lambda$. Then $z = f(x, y)$ is a Monge patch and by Proposition 3.1, it is a regular surface. □

Example 3.3.

$$F = x^2 + y^2 + z^2 \implies \text{grad}(F) = 2(x, y, z).$$

$r^2 \neq 0$ is a regular value but 0 is not.

Example 3.4.

$$G = x^2 + y^2 - z^2 \implies \text{grad}(G) = 2(x, y, -z).$$

$a \neq 0$ is a regular value but 0 is not. Actually, $G^{-1}(0)$ is a cone, which is not a regular surface at the vertex.

Proposition 3.3. *If there are two coordinate maps $\boldsymbol{r}_i : U_i \to V_i \cap S$ locally at p, then $\boldsymbol{r}_2^{-1} \circ \boldsymbol{r}_1 : \boldsymbol{r}_1^{-1}(V_1 \cap V_2 \cap S) \to \boldsymbol{r}_2^{-1}(V_1 \cap V_2 \cap S)$ is a smooth bijection with inverse.*

Proof.

$$\left(\frac{\partial(x, y)}{\partial(u_1, v_1)}, \frac{\partial(y, z)}{\partial(u_1, v_1)}, \frac{\partial(z, x)}{\partial(u_1, v_1)}\right) = \boldsymbol{r}_x \times \boldsymbol{r}_y \neq 0 \implies \text{WLOG,} \quad \frac{\partial(x, y)}{\partial(u_1, v_1)} \neq 0.$$

By the inverse function theorem, there exists $u_1 = f(x, y), v_1 = g(x, y)$.

$$\frac{\partial(u_1, v_1)}{\partial(u_2, v_2)} = \frac{\partial(u_1, v_1)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u_2, v_2)} = \left(\frac{\partial(x, y)}{\partial(u_1, v_1)}\right)^{-1} \frac{\partial(x, y)}{\partial(u_2, v_2)}.$$

Hence $(u_2, v_2) \to (u_1, v_1)$ is differentiable and $(u_1, v_1) \to (u_2, v_2)$ is differentiable.

□