Electrodynamics

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Abstract

This note is based on Electrodynamics B, taught by Mingzhi Li in Fall 2025 at Peking University, and also refers to Electrodynamics by Shuohong Guo.

Contents

1 Universal Laws of Electromagnetic Phenomena

1.1 Electric Charges and Electric Fields

Assume that the charges continuously distributed within V, then we have

$$\mathrm{d}Q = \rho(\boldsymbol{x'})\mathrm{d}V'$$

And the electric field at $\boldsymbol{x}$ is

$$\boldsymbol{E}(\boldsymbol{x}) = \int_V \frac{\rho(\boldsymbol{x'})}{4\pi\varepsilon_0 \boldsymbol{r}^3}\boldsymbol{r}\mathrm{d}V'$$

First recall the operator ∇:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

And we can define the gradient, divergence and curl as follows:

Definition 1.1.

- Gradient:

$$\mathrm{grad}\boldsymbol{f} = \nabla \boldsymbol{f} = \left(\frac{\partial f_x}{\partial x}, \frac{\partial f_y}{\partial y}, \frac{\partial f_z}{\partial z}\right)$$

- Divergence:

$$\mathrm{div}\boldsymbol{f} = \nabla \cdot \boldsymbol{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = \lim_{\Delta V \to 0} \frac{\oint \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S}}{\Delta V}$$

- Curl:

$$\mathrm{rot}\boldsymbol{f} = \nabla \times \boldsymbol{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \lim_{\Delta S \to 0} \frac{\oint \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{l}}{\Delta S}$$

Theorem 1.2 (Gauss Theorem)**.**

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0} \tag{1.1}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1.2}$$

Proof. Let Q denote the total charge enclosed by the surface, then the electric flux through the surface is

$$\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \cos\theta \mathrm{d}S$$

And $\frac{\cos\theta \mathrm{d}S}{r^2}$ is the solid angle $\mathrm{d}\Omega$ subtended by $\mathrm{d}\boldsymbol{S}$ at the charge. Therefore, we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{4\pi\varepsilon_0}\oint \mathrm{d}\Omega = \frac{Q}{\varepsilon_0}$$

In the continuous case, we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0}\int_V \rho \mathrm{d}V$$

Therefore, by the definition of divergence, we have

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

□

Theorem 1.3. *Electrostatic fields are irrotational, i.e.,*

$$\oint_L \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{1.3}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{1.4}$$

Proof. By Coulomb's law, we have

$$\oint_L \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0} \oint_L \frac{\boldsymbol{r}}{r^3} \cdot \mathrm{d}\boldsymbol{l} = \frac{Q}{4\pi\varepsilon_0} \oint_L \frac{\mathrm{d}r}{r^2} = \frac{Q}{4\pi\varepsilon_0} \oint_L \mathrm{d}\left(-\frac{1}{r}\right) = 0$$

By decreasing the loop to infinitesimal, we have

$$\nabla \times \boldsymbol{E} = 0$$

□

Proposition 1.4. *Theorem 1.3 holds whatever the equation of Coulomb's law is. It only requires the direction of the force (or field).*

Proof. Let $\boldsymbol{F} = f(r)\boldsymbol{r}$. Then, we have

$$\nabla \times \boldsymbol{F}|_{\hat{i}} = \frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}(f(r)y) = f'(r)\frac{yz}{r} - f'(r)\frac{yz}{r} = 0$$

□

1.2 Currents and Magnetic Fields

First, we recall Gauss theorem.

Theorem 1.5 (Gauss Theorem)**.**

$$\oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{F} \mathrm{d}V \tag{1.5}$$

Let $\boldsymbol{J}$ denote the current density, then we have

$$\mathrm{d}I = \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$$

And assume that there are several charged particles with charge density ρ_i moving with velocity $\boldsymbol{v_i}$, then we have

$$\boldsymbol{J} = \sum_i \rho_i \boldsymbol{v_i}$$

Theorem 1.6.

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1.6}$$

Particularly, in the steady case, we have $\frac{\partial \rho}{\partial t} = 0$, *thus*

$$\nabla \cdot \boldsymbol{J} = 0 \tag{1.7}$$

Proof. By the conservation of charge, we have

$$-\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V = \oint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$$

Therefore, by Gauss theorem 1.2, we have

$$\oint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{J} \mathrm{d}V$$

Since V is arbitrary, we have

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

□

Proposition 1.7 (Biot-Savart Law)**.**

$$\boldsymbol{B}(x) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}') \times \boldsymbol{r}}{r^3}\mathrm{d}V' \tag{1.8}$$

Then, the magnetic field $\boldsymbol{B}$ *at* $\boldsymbol{x}$ *due to a steady current* I *in a wire* L *is given by*

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int_L \frac{I\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3} \tag{1.9}$$

Proposition 1.8. *Magnetic fields are solenoidal because there is no isolated magnetic charge, i.e.,*

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0 \tag{1.10}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.11}$$

Proof. Notice that

$$\nabla \times \left(\frac{\boldsymbol{J}(\boldsymbol{x}')}{r}\right) = \left(\nabla\frac{1}{r}\right) \times \boldsymbol{J}(\boldsymbol{x}') + \frac{1}{r}\nabla \times \boldsymbol{J}(\boldsymbol{x}') = \frac{\boldsymbol{J}(\boldsymbol{x}') \times \boldsymbol{r}}{r^3}.$$

Denote $\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}')}{r}\mathrm{d}V'$, which is called the vector potential, then we have

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot (\nabla \times \boldsymbol{A}) = 0.$$

□

Theorem 1.9 (Amperé Circuital Theorem).

$$\oint_L \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I \tag{1.12}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{1.13}$$

Proof. Notice that

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}.$$

$$\nabla \cdot \boldsymbol{A} = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{x}')}{r} \right) \mathrm{d}V' = \frac{\mu_0}{4\pi} \int_V \left(\nabla \frac{1}{r} \cdot \boldsymbol{J}(\boldsymbol{x}') + \frac{1}{r} \nabla \cdot \boldsymbol{J}(\boldsymbol{x}') \right) \mathrm{d}V'.$$

Since $\nabla' \frac{1}{r} = -\nabla \frac{1}{r}$ and $\nabla \cdot \boldsymbol{J}(\boldsymbol{x}') = 0$, we have

$$\nabla \cdot \boldsymbol{A} = -\frac{\mu_0}{4\pi} \int_V \nabla' \frac{1}{r} \cdot \boldsymbol{J}(\boldsymbol{x}') \mathrm{d}V' = -\frac{\mu_0}{4\pi} \int_V \nabla' \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{x}')}{r} \right) \mathrm{d}V'.$$

By Gauss theorem, we have

$$\nabla \cdot \boldsymbol{A} = -\frac{\mu_0}{4\pi} \oint_S \frac{\boldsymbol{J}(\boldsymbol{x}')}{r} \cdot \mathrm{d}\boldsymbol{S}' = 0.$$

Therefore, we have

$$\nabla \times \boldsymbol{B} = -\nabla^2 \boldsymbol{A} = -\frac{\mu_0}{4\pi} \int_V \boldsymbol{J}(\boldsymbol{x}') \nabla^2 \frac{1}{r} \mathrm{d}V' = \frac{\mu_0}{4\pi} \int_V \boldsymbol{J}(\boldsymbol{x}') 4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') = \mu_0 \boldsymbol{J}(\boldsymbol{x}).$$

□

Example 1.10. Consider a long straight wire carrying a steady current I. By $I = \int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$, we have $\oint_L \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$. By the definition of curl, we have

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

1.3 Maxwell's Equations

Proposition 1.11 (Faraday's Law). *Induced electric field is a curl field , i.e.,*

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.14}$$

Proof. By the law of electromagnetic induction, we have

$$\mathscr{E} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

By $\mathscr{E} = \oint_L \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$, we have

$$\oint_L \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

By Stokes theorem and the arbitratiness of S, we have

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

□

We can show that Proposition 1.8 and Proposition 1.11 are consistent by

$$\nabla \cdot (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

However, notice that $\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 \nabla \cdot \boldsymbol{J} = 0$ does not hold in the non-steady case. Therefore, Maxwell added the term $\boldsymbol{J}_D$, called the **displacement current**, such that

$$\nabla \cdot (\boldsymbol{J} + \boldsymbol{J}_D) = 0.$$

By the continuity equation, we have

$$\nabla \cdot \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) = 0.$$

Therefore, we have

$$\boldsymbol{J}_D = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Theorem 1.12 (Maxwell's Equations)**.**

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1.15}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.16}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.17}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{1.18}$$

Hertz's experiments and the electromagnetic property of light verified Maxwell's equations.

The Lorentz force density and the Lorentz force acting on a particle is

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}, \quad \boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} \tag{1.19}$$

1.4 Electromagnetic Properties of Dielectrics

Definition 1.13. The **polarization $\boldsymbol{P}$** is the dipole moment per unit volume, i.e.,

$$\boldsymbol{P} = \lim_{\Delta V \to 0} \frac{\sum_i \boldsymbol{p}_i}{\Delta V}. \tag{1.20}$$

Consider $\mathrm{d}\boldsymbol{S}$ on the surface $\boldsymbol{S}$ in the dielectric, then we have the number of positive charges passing through $\mathrm{d}\boldsymbol{S}$ equals to

$$V \cdot nq = nq \cdot \boldsymbol{l} \cdot \mathrm{d}\boldsymbol{S} = n\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}$$

Definition 1.14. **Bound charges** are the charges induced by the polarization. Denote ρ_p and σ_p as the volume and surface bound charge densities, respectively.

$$\oint_S \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S} = -\int_V \rho_p \mathrm{d}V.$$

By Gauss theorem, we have

$$\nabla \cdot \boldsymbol{P} = -\rho_p.$$

Consider the boundary between two dielectrics, then we have

$$\sigma_p \mathrm{d}S = -(\boldsymbol{P}_2 - \boldsymbol{P}_1) \cdot \mathrm{d}\boldsymbol{S} = -(\boldsymbol{P}_2 - \boldsymbol{P}_1) \cdot \boldsymbol{n} \mathrm{d}S.$$

For Maxwell's equations in the dielectric, we have

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho_f + \rho_p = \rho_f - \nabla \cdot \boldsymbol{P}.$$

Definition 1.15. The **electric displacement $\boldsymbol{D}$** is defined as

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{1.21}$$

Then, we have

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{1.22}$$

For linear, isotropic and homogeneous dielectrics, we have

$$\boldsymbol{P} = \chi_e \varepsilon_0 \boldsymbol{E},$$

where χ_e is the electric susceptibility of the dielectric. Then, we have

$$\boldsymbol{D} = \varepsilon \boldsymbol{E},$$

where $\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0(1 + \chi_e)$ is the permittivity of the dielectric and ε_r is the relative permittivity of the dielectric.

If the molecular currents are viewed as tiny current loops with current i and area vector $\boldsymbol{a}$, then we have $\boldsymbol{m} = i\boldsymbol{a}$.

Definition 1.16. The **magnetization $\boldsymbol{M}$** is defined as the magnetic moment per unit volume, i.e.,

$$\boldsymbol{M} = \lim_{\Delta V \to 0} \frac{\sum_i \boldsymbol{m}_i}{\Delta V}. \tag{1.23}$$

Consider the boundary L of the surface S and compute the magnetization current I_M from back to front, then we have

$$I_M = i \oint_L n\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \oint_L n\boldsymbol{m} \cdot \mathrm{d}\boldsymbol{l} = \oint_L \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{l}.$$

By Stokes theorem, we have

$$I_M = \int_S \nabla \times \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{S}, \quad \boldsymbol{J}_M = \nabla \times \boldsymbol{M}, \tag{1.24}$$

where $\boldsymbol{J}_M$ is called the **magnetization current density**.

When the electric field changes, the polarization $\boldsymbol{P}$ changes, which causes another kind of current, called the **polarization current**. By $\boldsymbol{p} = e\boldsymbol{l} = \sum_i e_i \boldsymbol{x}_i$, we have

$$\boldsymbol{P} = \lim_{\Delta V \to 0} \frac{\sum_i \boldsymbol{p}_i}{\Delta V} = \lim_{\Delta V \to 0} \frac{\sum_i e_i \boldsymbol{x}_i}{\Delta V}, \quad \frac{\partial \boldsymbol{P}}{\partial t} = \lim_{\Delta V \to 0} \frac{\sum_i e_i \boldsymbol{v}_i}{\Delta V} = \boldsymbol{J}_P, \tag{1.25}$$

where $\boldsymbol{J}_P$ is called the **polarization current density**.

Consider the **total induced current density** $\boldsymbol{J} = \boldsymbol{J}_M + \boldsymbol{J}_P$ and free current density $\boldsymbol{J}_f$ in Maxwell's equations, we have

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B} = \boldsymbol{J}_f + \boldsymbol{J}_M + \boldsymbol{J}_P + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \quad \nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}.$$

Definition 1.17. The **magnetic field intensity** $\boldsymbol{H}$ is defined as

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}. \tag{1.26}$$

Then, we have

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1.27}$$

For isotropic, homogeneous and nonferromagnetic media, we have

$$\boldsymbol{M} = \chi_m \boldsymbol{H},$$

where χ_m is the magnetic susceptibility of the medium. Then, we have

$$\boldsymbol{B} = \mu \boldsymbol{H},$$

where $\mu = \mu_0 \mu_r = \mu_0(1 + \chi_m)$ is the permeability of the medium and μ_r is the relative permeability of the medium.

Theorem 1.18 (Maxwell's Equations in Media)**.**

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho_f \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \end{aligned} \tag{1.28}$$

Also, we have electromagnetic property equations of media

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{B} = \mu \cdot \boldsymbol{H}, \boldsymbol{J} = \sigma \cdot \boldsymbol{E},$$

For anisotropic media, the relations are more complex: ε is a three-dimensional second-order tensor

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

$$D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j, \quad i = 1, 2, 3.$$

Under strong fields, some media can be nonlinear, and we have the nonlinear relation:

$$D_i = \sum_j \varepsilon_{ij} E_j + \sum_{jk} \varepsilon_{ijk} E_j E_k + \sum_{jkl} \varepsilon_{ijkl} E_j E_k E_l + \cdots$$

For ferromagnetic materials, $\boldsymbol{B}$ and $\boldsymbol{H}$ are nonlinear and non-single-valued, which depends on the history of magnetization.

1.5 Boundary Conditions

Consider the normal component of $\boldsymbol{D}$. Applying the Maxwell equation $\oint_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = Q_f$ to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we have $(D_{2n} - D_{1n})\Delta S + \int_{\text{Side}} \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = \sigma_f \Delta S$. As $h \to 0$, we have $\int \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = 0$. Therefore, we have

$$D_{2n} - D_{1n} = \sigma_f. \tag{1.29}$$

It shows that the normal component of $\boldsymbol{D}$ is discontinuous across the boundary if there is free surface charge. Similarly, for $\boldsymbol{B}$, we have

$$B_{2n} - B_{1n} = 0. \tag{1.30}$$

Consider the tangent component of $\boldsymbol{H}$. Denote the density of free surface current as $\boldsymbol{\alpha}$. Applying the Maxwell equation $\oint_L \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = I_f + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S}$ to a very thin Ampèrian loop stradlling the surface, we have $(H_{2t} - H_{1t})\Delta l + \int \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{\alpha}_f \Delta l + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S}$. As $S \to 0$, we have $\int \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = 0$. Therefore, we have

$$(\boldsymbol{H}_2 - \boldsymbol{H}_1)_{//} = \boldsymbol{\alpha}_f \times \boldsymbol{e}_n, \quad \boldsymbol{e}_n \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{\alpha}_f. \tag{1.31}$$

It shows that the tangential component of $\boldsymbol{H}$ is discontinuous across the boundary if there is free surface current. Similarly, for $\boldsymbol{E}$, we have

$$(\boldsymbol{E}_2 - \boldsymbol{E}_1)_{//} = 0, \quad \boldsymbol{e}_n \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0. \tag{1.32}$$

Proposition 1.19. *The boundary conditions of electromagnetic fields are*

$$\begin{aligned} \boldsymbol{e}_n \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) &= 0 \\ \boldsymbol{e}_n \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) &= \boldsymbol{\alpha} \\ \boldsymbol{e}_n \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) &= \sigma \\ \boldsymbol{e}_n \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) &= 0 \end{aligned} \tag{1.33}$$

1.6 Energy and Energy Flow of Electromagnetic Fields

Definition 1.20. The **energy density** of electromagnetic fields is defined as the energy per unit volume, denoted as $w = w(\boldsymbol{x}, t)$. The **energy flux density** of electromagnetic fields is defined as the energy flowing through a unit area per unit time, denoted as $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{x}, t)$, also called the **Poynting vector**.

By the conservation of energy, we have

$$-\oint_S \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{\sigma} = \int_V \boldsymbol{f}\cdot \boldsymbol{v}\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_V w\mathrm{d}V, \quad \nabla\cdot \boldsymbol{S} + \frac{\partial w}{\partial t} = -\boldsymbol{f}\cdot \boldsymbol{v}$$

Consider V is the whole space, then we have

$$\int_\infty \boldsymbol{f}\cdot \boldsymbol{v}\mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\infty w\mathrm{d}V = 0. \tag{1.34}$$

By the Lorentz force

$$\boldsymbol{f}\cdot \boldsymbol{v} = (\rho \boldsymbol{E} + \rho \boldsymbol{v}\times \boldsymbol{B})\cdot \boldsymbol{v} = \boldsymbol{J}\cdot \boldsymbol{E}$$

and the Maxwell equations, we have

$$\boldsymbol{J} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}.$$

Therefore, we have the formula of energy flux density and the change rate of energy density of electromagnetic fields

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \quad \frac{\partial w}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{B} \cdot \frac{\partial \boldsymbol{H}}{\partial t}. \tag{1.35}$$

In vacuum, we have $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$ and $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, thus

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}, \quad w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}) = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2). \tag{1.36}$$

In linear media, we have $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, thus

$$w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}). \tag{1.37}$$

2 Electrostatic

2.1 Potential and its Differential Equations

Because $\nabla \times \boldsymbol{E} = 0$, $\oint_L \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$ holds for any loop, $\int_{P_1}^{P_2} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$ is path-independent.

Definition 2.1. The **electric potential difference** V between two points P_1 and P_2 is defined as the work done by an external agent in bringing a unit positive charge from P_1 to P_2, i.e.,

$$\varphi(P_2) - \varphi(P_1) = -\int_{P_1}^{P_2} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \tag{2.1}$$

Since $\mathrm{d}\varphi = -\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$ and $\mathrm{d}\varphi = \frac{\partial \varphi}{\partial x}\mathrm{d}x + \frac{\partial \varphi}{\partial y}\mathrm{d}y + \frac{\partial \varphi}{\partial z}\mathrm{d}z = \nabla\varphi \cdot \mathrm{d}\boldsymbol{l}$, we have

$$\boldsymbol{E} = -\nabla\varphi. \tag{2.2}$$

We often set the potential at infinity as zero, i.e., $\varphi(\infty) = 0$. Therefore, we have

$$\varphi(P) = \int_P^\infty \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}. \tag{2.3}$$

Given a point charge Q, $\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\boldsymbol{e}_r$, thus

$$\varphi(P) = \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{Q}{r'^2}\mathrm{d}r' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}. \tag{2.4}$$

For a continuous distribution of charge, we have

$$\varphi(P) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\boldsymbol{x}')}{r}\mathrm{d}V'. \tag{2.5}$$

For linear, isotropic and homogeneous dielectrics, we have **Poisson's Equation**

$$\rho = \nabla \cdot \boldsymbol{D} = \nabla \cdot (\varepsilon\boldsymbol{E}) = -\varepsilon\nabla \cdot (\nabla\varphi) = -\varepsilon\nabla^2\varphi \Longrightarrow \nabla^2\varphi = -\frac{\rho}{\varepsilon}. \tag{2.6}$$

Now we discuss the boundary conditions. By definition, we have $\varphi_1 = \varphi_2$ on the two sides of the boundary. Consider P_1 and P_1', P_2 and P_2' are two pairs of points that are very close to each other and on the opposite sides of the boundary, respectively. Then, we have $\varphi_1' - \varphi_1 = \varphi_2' - \varphi_2$ and $\boldsymbol{E}_1 \cdot \Delta\boldsymbol{l} = \boldsymbol{E}_2 \cdot \Delta\boldsymbol{l}$. Hence $\boldsymbol{E}_{2//} = \boldsymbol{E}_{1//}$.

Another boundary condition can be written as

$$\sigma_f = \boldsymbol{n}\cdot(\boldsymbol{D}_2 - \boldsymbol{D}_1) = \boldsymbol{n}\cdot(\varepsilon_2\boldsymbol{E}_2 - \varepsilon_1\boldsymbol{E}_1) = \varepsilon_2\boldsymbol{n}\cdot(-\nabla\varphi_2) - \varepsilon_1\boldsymbol{n}\cdot(-\nabla\varphi_1) = -\varepsilon_2\frac{\partial\varphi_2}{\partial n} + \varepsilon_1\frac{\partial\varphi_1}{\partial n}. \tag{2.7}$$

For a conductor, $\rho = 0$, $\boldsymbol{E} = 0$ and $\varphi =$ constant inside the conductor. And $\boldsymbol{E}$ is perpendicular to the surface of the conductor, hence the conductor is an equipotential body. The boundary conditions on the surface of the conductor are

$$\varphi = \text{constant}, \quad \varepsilon\frac{\partial\varphi}{\partial n} = -\sigma. \tag{2.8}$$

The energy stored in the electrostatic field in volume V is

$$\begin{aligned} W &= \frac{1}{2}\int_\infty \boldsymbol{E}\cdot\boldsymbol{D}\mathrm{d}V = \frac{1}{2}\int_\infty -\nabla\varphi\cdot\boldsymbol{D}\mathrm{d}V = \frac{1}{2}\int_\infty \left(-\nabla\cdot(\varphi\boldsymbol{D}) + \varphi\nabla\cdot\boldsymbol{D}\right)\mathrm{d}V \\ &= \frac{1}{2}\int_\infty \left(-\nabla\cdot(\varphi\boldsymbol{D}) + \rho\varphi\right)\mathrm{d}V = -\frac{1}{2}\oint_S \varphi\boldsymbol{D}\cdot\mathrm{d}\boldsymbol{S} + \frac{1}{2}\int_\infty \rho\varphi\mathrm{d}V = \frac{1}{2}\int_\infty \rho\varphi\mathrm{d}V. \end{aligned} \tag{2.9}$$

For charge distribution ρ, we have

$$W = \frac{1}{8\pi\varepsilon}\int_\infty\int_\infty \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{x}')}{r}\mathrm{d}V'\mathrm{d}V. \tag{2.10}$$

2.2 Uniqueness Theorem

Theorem 2.2 (First Uniqueness Theorem)**.** *Consider there are n dielectrics V_i in V and the free charge density is ρ. The electric field in V is unique if either the potential $\varphi|_S$ is given, or $\frac{\partial\varphi}{\partial n}\big|_S$ is given on the boundary S.*

Proof. First, we have Poisson's equation $\nabla^2\varphi = -\frac{\rho}{\varepsilon}$ and boundary conditions $\varphi_i = \varphi_j$, $\varepsilon_i\left(\frac{\partial\varphi}{\partial n}\right)_i = \varepsilon_j\left(\frac{\partial\varphi}{\partial n}\right)_j$.

Suppose there are two solutions φ' and φ'', and denote $\varphi = \varphi' - \varphi''$. Then we have $\nabla^2\varphi = 0$ in every V_i and $\varphi|_S = 0$, $\frac{\partial\varphi}{\partial n}\big|_S = 0$. Consider the integral on each surface of V_i:

$$\oint_{S_i} \varepsilon_i\varphi\nabla\varphi\cdot\mathrm{d}\boldsymbol{S} = \int_{V_i}\varepsilon_i\nabla\cdot(\varphi\nabla\varphi)\mathrm{d}V = \int_{V_i}\varepsilon_i(\varphi\nabla^2\varphi + (\nabla\varphi)^2)\mathrm{d}V = \int_{V_i}\varepsilon_i(\nabla\varphi)^2\mathrm{d}V.$$

Since the normal component of φ and $\varepsilon\nabla\varphi$ are eqaul on the boundary of two dielectrics, and $\varphi = 0$ or $\frac{\partial\varphi}{\partial n} = 0$ on the surface S, we have

$$\sum_i\int_{V_i}\varepsilon_i(\nabla\varphi)^2\mathrm{d}V = 0.$$

Therefore, $\nabla\varphi = 0$ and $\varphi =$ constant. □

Theorem 2.3 (Second Uniqueness Theorem)**.** *Consider there are n dielectrics V_i in V and the free charge density is ρ. The electric field in V is unique if either the potential $\varphi|_S$ is given, or $\frac{\partial\varphi}{\partial n}\big|_S$ is given on the boundary S, and the total charge on each conductor inside V is given.*

2.3 Laplace's Equation

First recall the Legendre polynomial.

Definition 2.4. $P_n(x)$ is called the **Legendre polynomial** of degree n if it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}\right] + n(n+1)P_n(x) = 0. \tag{2.11}$$

Generally, $P_n(x)$ can be expressed as

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n. \tag{2.12}$$

For example, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$.

We are interested in the solution when $\rho = 0$, i.e., Laplace's equation

$$\nabla^2 \varphi = 0. \tag{2.13}$$

In Cartesian coordinates, we have

$$\varphi(x, y, z) = \left(C_1 e^{\alpha_1 x} + C_2 e^{-\alpha_1 x}\right)\left(C_3 e^{\alpha_2 y} + C_4 e^{-\alpha_2 y}\right)\left(C_5 e^{\alpha_3 z} + C_6 e^{-\alpha_3 z}\right). \tag{2.14}$$

In spherical coordinates, we have

$$\begin{aligned} \varphi(r, \theta, \phi) = & \sum_{n,m} \left(a_{nm} R^n + \frac{b_{nm}}{R^{n+1}}\right) P_n^m(\cos\theta)\cos(m\phi) \\ & + \sum_{n,m} \left(c_{nm} R^n + \frac{d_{nm}}{R^{n+1}}\right) P_n^m(\cos\theta)\sin(m\phi), \end{aligned} \tag{2.15}$$

where $P_n^m(x)$ is the associated Legendre function. If there is a symmetry about the azimuthal angle ϕ, then we have

$$\varphi(r, \theta) = \sum_n \left(a_n R^n + \frac{b_n}{R^{n+1}}\right) P_n(\cos\theta). \tag{2.16}$$

Example 2.5. A conductor ball of radius R is held at zero potential and surrounded by a spherical conductor surface of inner radius R_2 and outer radius R_1 with charge Q. Find the potential in the space and induced charge on the surface of the ball.

Proof. By symmetry, we have φ only depends on R. Therefore, we have

$$\varphi_1 = a + \frac{b}{R}, \quad R > R_3 \text{ and } \varphi_2 = c + \frac{d}{R}, \quad R_1 < R < R_2.$$

We have boundary conditions

$$\varphi_2|_{R=R_1} = \varphi_1|_{R\to\infty} = 0, \quad \varphi_2|_{R=R_2} = \varphi_1|_{R=R_3}, \quad -\oint_{R=R_3} \frac{\partial \varphi_1}{\partial R} R^2 \mathrm{d}\Omega + \oint_{R=R_2} \frac{\partial \varphi_2}{\partial R} R^2 \mathrm{d}\Omega = \frac{Q}{\varepsilon_0}.$$

Therefore, we have

$$\varphi_1 = \frac{Q + Q_1}{4\pi\varepsilon_0 R}, \quad \varphi_2 = \frac{Q_1}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R_1}\right),$$

where $Q_1 = -\frac{R_3^{-1}}{R_1^{-1} - R_2^{-1} + R_3^{-1}} Q$. The induced charge on the surface of the ball is

$$-\varepsilon_0 \oint_{R=R_1} \frac{\partial \varphi_2}{\partial R} R^2 \mathrm{d}\Omega = Q_1.$$

□

Example 2.6. A dielectric ball of permittivity ε is placed in a uniform electric field $\boldsymbol{E}_0$. Find the potential.

Proof. Denote the potential outside the ball as φ_1 and inside the ball as φ_2. By symmetry, we have

$$\varphi_1 = \sum_n \left(a_n R^n + \frac{b_n}{R^{n+1}} \right) P_n(\cos\theta), \quad \varphi_2 = \sum_n c_n R^n P_n(\cos\theta).$$

We have boundary conditions at infinity

$$\varphi_1 \to -E_0 R\cos\theta = -E_0 R P_1(\cos\theta) \Longrightarrow a_1 = -E_0, \quad a_n = 0 (n \neq 1).$$

φ_2 is finite when $R = 0$, hence $b_n = 0$. On the surface of the ball, we have

$$\varphi_1 = \varphi_2, \quad \varepsilon_0 \frac{\partial \varphi_1}{\partial R} = \varepsilon \frac{\partial \varphi_2}{\partial R}.$$

Therefore, we have

$$\varphi_1 = -E_0 R\cos\theta + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \frac{R_0^3}{R^2} \cos\theta, \quad \varphi_2 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 R\cos\theta.$$

□

2.4 The Method of Images

Example 2.7. A point charge Q is held a distance d above an infinite grounded conducting plane. Find the potential in the half-space above the plane.

Proof. We have boundary conditions $\varphi =$ constant. □

Example 2.8. A point charge Q is held a distance a from the center of a conducting sphere of radius R_0 ($a > R_0$). Find the potential outside the sphere.

Proof. Imagine a point charge Q' is placed on the line connecting the center of the sphere and Q, such that the potential on the surface of the sphere is zero, i.e. $\frac{Q}{r} + \frac{Q'}{R_0} = 0$.

$$\varphi = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{r} - \frac{R_0 Q'}{a r'} \right)$$

□

2.5 Green Function

2.6 Multipole Expansion

We introduce some concepts of tensor and dyadic.

Definition 2.9. Let $\boldsymbol{A} = A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3$, $\boldsymbol{B} = B_1\boldsymbol{e}_1 + B_2\boldsymbol{e}_2 + B_3\boldsymbol{e}_3$, then the **dyadic**

$$\begin{aligned}\overleftrightarrow{T} = \boldsymbol{AB} &= A_1B_1\boldsymbol{e}_1\boldsymbol{e}_1 + A_1B_2\boldsymbol{e}_1\boldsymbol{e}_2 + A_1B_3\boldsymbol{e}_1\boldsymbol{e}_3 + A_2B_1\boldsymbol{e}_2\boldsymbol{e}_1 + A_2B_2\boldsymbol{e}_2\boldsymbol{e}_2 + A_2B_3\boldsymbol{e}_2\boldsymbol{e}_3 \\ &+ A_3B_1\boldsymbol{e}_3\boldsymbol{e}_1 + A_3B_2\boldsymbol{e}_3\boldsymbol{e}_2 + A_3B_3\boldsymbol{e}_3\boldsymbol{e}_3 = [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j.\end{aligned} \tag{2.17}$$

In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. If $T_{ij} = T_{ji}$, then T is called a **symmetric tensor**. If $T_{ij} = -T_{ji}$, then T is called a **antisymmetric tensor**. $\overleftrightarrow{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the **unit tensor**.

All the nine $\boldsymbol{e}_i\boldsymbol{e}_j$ form the basis of this tensor space.

For vector $\boldsymbol{f}$, the dot product is defined as $\boldsymbol{f} \cdot \boldsymbol{AB} = (\boldsymbol{f} \cdot \boldsymbol{A})\boldsymbol{B}$ and the cross product is defined as $\boldsymbol{f} \times \boldsymbol{AB} = (\boldsymbol{f} \times \boldsymbol{A})\boldsymbol{B}$. In general, $\boldsymbol{f} \cdot \boldsymbol{AB} \neq \boldsymbol{AB} \cdot \boldsymbol{f}$ and $\boldsymbol{f} \times \boldsymbol{AB} \neq \boldsymbol{AB} \times \boldsymbol{f}$.

We can also define that $(\boldsymbol{AB}) \cdot (\boldsymbol{CD}) = (\boldsymbol{B} \cdot \boldsymbol{C})(\boldsymbol{A} \cdot \boldsymbol{D})$ and $(\boldsymbol{AB}) : (\boldsymbol{CD}) = (\boldsymbol{B} \cdot \boldsymbol{C})(\boldsymbol{A} \cdot \boldsymbol{D})$, hence $(\boldsymbol{AB}) : (\boldsymbol{CD}) = (\boldsymbol{CD}) : (\boldsymbol{AB})$.

Proposition 2.10.

$$\nabla \cdot \overleftrightarrow{T} = \frac{\partial}{\partial x}\left(\boldsymbol{e}_x \cdot \overleftrightarrow{T}\right) + \frac{\partial}{\partial y}\left(\boldsymbol{e}_y \cdot \overleftrightarrow{T}\right) + \frac{\partial}{\partial z}\left(\boldsymbol{e}_z \cdot \overleftrightarrow{T}\right),$$

$$\begin{aligned}\nabla \times \overleftrightarrow{T} = \boldsymbol{e}_x &\left(\frac{\partial}{\partial y}\left(\boldsymbol{e}_z \cdot \overleftrightarrow{T}\right) - \frac{\partial}{\partial z}\left(\boldsymbol{e}_y \cdot \overleftrightarrow{T}\right)\right) + \boldsymbol{e}_y\left(\frac{\partial}{\partial z}\left(\boldsymbol{e}_x \cdot \overleftrightarrow{T}\right) - \frac{\partial}{\partial x}\left(\boldsymbol{e}_z \cdot \overleftrightarrow{T}\right)\right) \\ &+ \boldsymbol{e}_z\left(\frac{\partial}{\partial x}\left(\boldsymbol{e}_y \cdot \overleftrightarrow{T}\right) - \frac{\partial}{\partial y}\left(\boldsymbol{e}_x \cdot \overleftrightarrow{T}\right)\right),\end{aligned}$$

$$\overleftrightarrow{T} \cdot \nabla = \left(\overleftrightarrow{T} \cdot \boldsymbol{e}_x\right)\frac{\partial}{\partial x} + \left(\overleftrightarrow{T} \cdot \boldsymbol{e}_y\right)\frac{\partial}{\partial y} + \left(\overleftrightarrow{T} \cdot \boldsymbol{e}_z\right)\frac{\partial}{\partial z}, \quad \nabla \cdot (\varphi\overleftrightarrow{T}) = \varphi\nabla \cdot \overleftrightarrow{T} + \nabla\varphi \cdot \overleftrightarrow{T}.$$

Since $\boldsymbol{x}' \cdot \nabla' = x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}$, $\nabla'\nabla' = \boldsymbol{e}_x\boldsymbol{e}_x\frac{\partial^2}{\partial x'^2} + \boldsymbol{e}_y\boldsymbol{e}_y\frac{\partial^2}{\partial y'^2} + \boldsymbol{e}_z\boldsymbol{e}_z\frac{\partial^2}{\partial z'^2} + \boldsymbol{e}_x\boldsymbol{e}_y\frac{\partial^2}{\partial x'\partial y'} + \boldsymbol{e}_y\boldsymbol{e}_z\frac{\partial^2}{\partial y'\partial z'} + \boldsymbol{e}_z\boldsymbol{e}_x\frac{\partial^2}{\partial z'\partial x'} + \boldsymbol{e}_y\boldsymbol{e}_x\frac{\partial^2}{\partial y'\partial x'} + \boldsymbol{e}_z\boldsymbol{e}_y\frac{\partial^2}{\partial z'\partial y'} + \boldsymbol{e}_x\boldsymbol{e}_z\frac{\partial^2}{\partial x'\partial z'}$ and $\boldsymbol{x}'\boldsymbol{x}' : \nabla'\nabla' = x'^2\frac{\partial^2}{\partial x'^2} + y'^2\frac{\partial^2}{\partial y'^2} + z'^2\frac{\partial^2}{\partial z'^2} + 2xy\frac{\partial^2}{\partial x'\partial y'} + 2yz\frac{\partial^2}{\partial y'\partial z'} + 2zx\frac{\partial^2}{\partial z'\partial x'}$, we have the Taylor's formula of $f(x', y', z')$:

$$f(x', y', z') = f(0) + (\boldsymbol{x}' \cdot \nabla')f(0) + \frac{1}{2!}(\boldsymbol{x}'\boldsymbol{x}' : \nabla'\nabla')f(0) + \cdots, \tag{2.18}$$

and

$$\begin{aligned}\frac{1}{r} &= \frac{1}{r}\bigg|_{\boldsymbol{x}'=0} + (\boldsymbol{x}' \cdot \nabla')\frac{1}{r}\bigg|_{\boldsymbol{x}'=0} + \frac{1}{2!}(\boldsymbol{x}'\boldsymbol{x}' : \nabla'\nabla')\frac{1}{r}\bigg|_{\boldsymbol{x}'=0} + \cdots \\ &= \left(\frac{1}{r} - (\boldsymbol{x} \cdot \nabla)\frac{1}{r} + \frac{1}{2!}(\boldsymbol{x}\boldsymbol{x} : \nabla\nabla)\frac{1}{r} + \cdots\right)\bigg|_{\boldsymbol{x}'=0} \\ &= \frac{1}{R} - \boldsymbol{x}' \cdot \nabla\frac{1}{R} + \frac{1}{2!}(\boldsymbol{x}'\boldsymbol{x}' : \nabla\nabla)\frac{1}{R} + \cdots,\end{aligned}$$

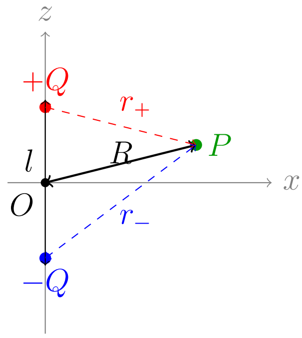
Therefore, denote $Q = \int_{V'}\rho(\boldsymbol{x}')\mathrm{d}V'$, $\boldsymbol{p} = \int_{V'}\rho(\boldsymbol{x}')\boldsymbol{x}'\mathrm{d}V'$ and $\mathscr{D} = \int_{V'} 3\rho(\boldsymbol{x}')\boldsymbol{x}'\boldsymbol{x}'\mathrm{d}V'$, then we have

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{x}')}{r}\mathrm{d}V' = \frac{1}{4\pi\varepsilon_0}\int\rho(\boldsymbol{x}')\left(\frac{1}{R} - \boldsymbol{x}' \cdot \nabla\frac{1}{R} + \frac{1}{2!}(\boldsymbol{x}'\boldsymbol{x}' : \nabla\nabla)\frac{1}{R} + \cdots\right)\mathrm{d}V' \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} - \boldsymbol{p} \cdot \nabla\frac{1}{R} + \frac{1}{6}\frac{\mathscr{D} : \nabla\nabla}{R} + \cdots\right) = \varphi^{(0)}(\boldsymbol{x}) + \varphi^{(1)}(\boldsymbol{x}) + \varphi^{(2)}(\boldsymbol{x}) + \cdots.\end{aligned} \tag{2.19}$$

Q is called the charge or monopole moment, $\boldsymbol{p}$ is called the electric dipole moment and $\mathscr{D}$ is called the electric quadrupole moment.

$\varphi^{(0)}$ is the potential of a point charge Q at the origin, $\varphi^{(1)}$ is the potential of a dipole $\boldsymbol{p}$, and $\varphi^{(2)}$ is the potential of a quadrupole $\mathscr{D}$. Denote $Q_{ij\cdots s} = \int\rho(\boldsymbol{x}')x'_i x'_j \cdots x'_s\mathrm{d}V'$ and this is called the multipole moment of order n.

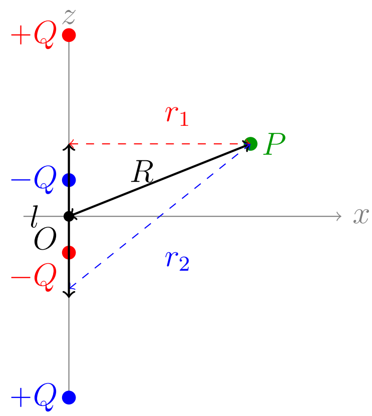
For dipole $\boldsymbol{p}$, we have $\boldsymbol{p} = \int_V \rho(\boldsymbol{x}')\boldsymbol{x}'\mathrm{d}V'$ or $\boldsymbol{p} = \sum_i q_i \boldsymbol{x}_i$. If the charge distribution is symmetric about the origin, then $\boldsymbol{p} = 0$.



In this example, $\boldsymbol{p} = Q\boldsymbol{l}$ and if $l \ll R$, we have

$$\begin{aligned}\varphi &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \approx \frac{Q}{4\pi\varepsilon_0}\frac{l\cos\theta}{R^2} = -\frac{Q}{4\pi\varepsilon_0} l \frac{\partial}{\partial z}\left(\frac{1}{R}\right)\\ &= -\frac{1}{4\pi\varepsilon_0} p_z \frac{\partial}{\partial z}\left(\frac{1}{R}\right) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{p}\cdot\nabla\frac{1}{R},\end{aligned}$$

which satisfies the previous result.



In this example, $Q = 0$ and $\boldsymbol{p} = 0$. Since $\mathscr{D}_{ij} = \int_V 3x_i'x_j'\rho(\boldsymbol{x}')\mathrm{d}V'$, we have $\mathscr{D}_{33} = 3(b^2 - a^2)2Q = 6pl$ and other components are zero. Hence

$$\begin{aligned}\varphi &\approx -\frac{1}{4\pi\varepsilon_0} p \frac{\partial}{\partial z}\left(\frac{1}{r_+}\right) + \frac{1}{4\pi\varepsilon_0} p \frac{\partial}{\partial z}\left(\frac{1}{r_-}\right) = -\frac{1}{4\pi\varepsilon_0} pl \frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\mathscr{D}_{33}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\end{aligned} \tag{2.20}$$

Proposition 2.11.

$$\nabla\nabla\frac{1}{R} = -\nabla\frac{\boldsymbol{R}}{R^3} = -\left(\left(\nabla\frac{1}{R^3}\right)\boldsymbol{R} + \frac{1}{R^3}\nabla\boldsymbol{R}\right). \tag{2.21}$$

Proof. □

That's because $\nabla\boldsymbol{r} = I$ and $\nabla\frac{1}{R^3} = -\frac{3\boldsymbol{R}}{R^5}$, we have

$$\begin{aligned}\varphi^{(2)}(\boldsymbol{x}) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\mathscr{D}:\nabla\nabla\frac{1}{R} = \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\int_{V'}3\rho(\boldsymbol{x}')\boldsymbol{x}'\boldsymbol{x}'\mathrm{d}V' : \left(\nabla\nabla\frac{1}{R}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{6}\left(\int_{V'}3\rho(\boldsymbol{x}')\boldsymbol{x}'\boldsymbol{x}'\mathrm{d}V' : \left(\nabla\nabla\frac{1}{R}\right) - \int\rho(\boldsymbol{x}')r'\mathrm{d}V'\right)\end{aligned} \tag{2.22}$$

If the charge distribution is spherically symmetric, then $\mathscr{D} = 0$.

$$\int_V \boldsymbol{x}'^2\rho(\boldsymbol{x}')$$

The ennergy of the charge distribution in the electrostatic field is

$$\begin{aligned}W &= \int\rho\varphi_e\mathrm{d}V = \int\rho(\boldsymbol{x})\left(\varphi_e(0) + (\boldsymbol{x}\cdot\nabla)\varphi_e(0) + \frac{1}{2!}(\boldsymbol{x}\boldsymbol{x}:\nabla\nabla)\varphi_e(0) + \cdots\right)\mathrm{d}V\\ &= Q\varphi_e(0) + \boldsymbol{p}\cdot\nabla\varphi_e(0) + \frac{1}{6}\mathscr{D}:\nabla\nabla\varphi_e(0) + \cdots,\end{aligned} \tag{2.23}$$

where $W^{(0)} = Q\varphi_e(0)$ is the energy of a point charge Q at the origin, $W^{(1)} = \boldsymbol{p}\cdot\nabla\varphi_e(0) = -\boldsymbol{p}\cdot\boldsymbol{E}_e(0)$ is the energy of a dipole $\boldsymbol{p}$ in the external field, and $W^{(2)} = \frac{1}{6}\mathscr{D}:\nabla\boldsymbol{E}_e(0)$ is the energy of the quadrupole in the external fields.

The force on the dipole is given by

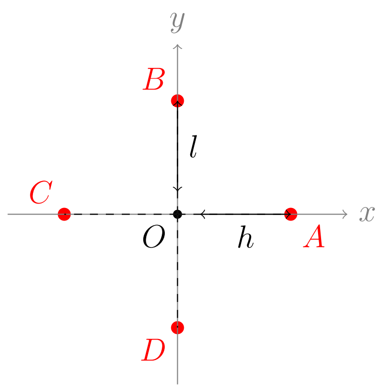
$$\boldsymbol{F} = -\nabla W^{(1)} = \nabla(\boldsymbol{p}\cdot\boldsymbol{E}_e) = \boldsymbol{p}\cdot\nabla\boldsymbol{E}_e, \tag{2.24}$$

and the torque is given by

$$L_\theta = -\frac{\partial W^{(1)}}{\partial\theta} = \frac{\partial}{\partial\theta}(pE_e\cos\theta) = -pE_e\sin\theta, \quad \boldsymbol{L} = \boldsymbol{p}\times\boldsymbol{E}_e. \tag{2.25}$$

2.7 Finite-Difference Method

Consider $\nabla^2\varphi = \frac{\partial^2}{\partial x^2}\varphi + \frac{\partial^2}{\partial y^2}\varphi = -\frac{\rho}{\varepsilon_0}$ in a region. Take any O in the region and let A, B, C, D be four points around O such that $OA = OC = h, OB = OD = l$.



By Taylor's expansion, we have

$$\varphi_A = \varphi_O + h\left(\frac{\partial\varphi}{\partial x}\right)\bigg|_O + \frac{h^2}{2}\left(\frac{\partial^2\varphi}{\partial x^2}\right)\bigg|_O + \cdots,$$
$$\varphi_B = \varphi_O + l\left(\frac{\partial\varphi}{\partial y}\right)\bigg|_O + \frac{l^2}{2}\left(\frac{\partial^2\varphi}{\partial y^2}\right)\bigg|_O + \cdots,$$
$$\varphi_C = \varphi_O - h\left(\frac{\partial\varphi}{\partial x}\right)\bigg|_O + \frac{h^2}{2}\left(\frac{\partial^2\varphi}{\partial x^2}\right)\bigg|_O + \cdots,$$
$$\varphi_D = \varphi_O - l\left(\frac{\partial\varphi}{\partial y}\right)\bigg|_O + \frac{l^2}{2}\left(\frac{\partial^2\varphi}{\partial y^2}\right)\bigg|_O + \cdots,$$

hence

$$\frac{\partial^2\varphi}{\partial x^2}\bigg|_O = \frac{\varphi_A - 2\varphi_O + \varphi_C}{h^2} + o(h^2), \quad \frac{\partial^2\varphi}{\partial y^2}\bigg|_O = \frac{\varphi_B - 2\varphi_O + \varphi_D}{l^2} + o(l^2).$$

Let $h = l$ and $h, l \to 0$, then we have

$$\varphi_O = \frac{1}{4}\left(\varphi_A + \varphi_B + \varphi_C + \varphi_D + h^2\frac{\rho_O}{\varepsilon_0}\right).$$

If $\rho = 0$, then $\varphi_O = \frac{1}{4}(\varphi_A + \varphi_B + \varphi_C + \varphi_D)$. This is called the **mean difference form of five points with equal spacing**.

For the first boundary value problem, the number of equations is equal to the number of interior points. For the second boundary value problem with boundary condition $\frac{\partial\varphi}{\partial n}\big|_O = k$, if C is outside the region, we have $\frac{\partial\varphi}{\partial n}\big|_O = \frac{\varphi_A - \varphi_C}{2h} = k$. Hence we have

$$\varphi_O = \frac{1}{4}\left(2\varphi_A + \varphi_B + \varphi_D - 2hk\right).$$

And the number of equations is also equal to the number of grid points.

We can use the **iterative method**, the **relaxation method (Gauss-Seidel)** and **successive over-relaxation method (SOR)** to solve these equations.

3 Magnetostatics

3.1 Vector Potential

For $\nabla \times \boldsymbol{f} = 0$, we have $\boldsymbol{f} = \nabla\varphi$ for some φ. For $\nabla \cdot \boldsymbol{f} = 0$, we have $\boldsymbol{f} = \nabla \times \boldsymbol{A}$ for some $\boldsymbol{A}$.

Definition 3.1. The **magnetic vector potential $\boldsymbol{A}$** is defined as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

$\boldsymbol{A}$ is not unique. Let $\boldsymbol{A}' = \boldsymbol{A} + \nabla\psi$, then $\nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} + \nabla \times (\nabla\psi) = \nabla \times \boldsymbol{A} = \boldsymbol{B}$. This is called the **gauge transformation**.

An irrotational field is also called a **longitudinal field $\boldsymbol{A}_l$** and a solenoidal field is also called a **transverse field $\boldsymbol{A}_t$**. Any vector field can be decomposed into the sum of an irrotational field and a solenoidal field, i.e. $\boldsymbol{A} = \boldsymbol{A}_l + \boldsymbol{A}_t$, then $\boldsymbol{B} = \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{A}_t$. Hence we can choose any $\boldsymbol{A}_l$. Then we have the gauge condition

$$\nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{A}_l = 0.$$

For S_1 and S_2 with the same boundary curve L, we have

$$\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_L \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \Longrightarrow \int_{S_1} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int_{S_2} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

This shows the solenoidality of $\boldsymbol{B}$. And the circulation of $\boldsymbol{A}$ is the magnetic flux through any surface bounded by L.

Proposition 3.2. *Since $\nabla \times \boldsymbol{B} = \mu\boldsymbol{J}$ and $\nabla \cdot \boldsymbol{A} = 0$, we have*

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu\boldsymbol{J} \Longrightarrow \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} = \mu\boldsymbol{J} \Longrightarrow \nabla^2\boldsymbol{A} = -\mu\boldsymbol{J}. \tag{3.1}$$

This is called the ***vector Poisson's equation****. In Cartesian coordinates, we have*

$$\nabla^2 A_i = -\mu J_i, \quad i = 1, 2, 3.$$

In cylindrical coordinates, we have

$$\begin{aligned}
\left(\nabla^2 A\right)_\rho &= \nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi}, \\
\left(\nabla^2 A\right)_\phi &= \nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi}, \\
\left(\nabla^2 A\right)_z &= \nabla^2 A_z.
\end{aligned}$$

In spherical coordinates, we have

$$\begin{aligned}
\left(\nabla^2 A\right)_r &= \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \\
\left(\nabla^2 A\right)_\theta &= \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}, \\
\left(\nabla^2 A\right)_\phi &= \nabla^2 A_\phi + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi} - \frac{A_\phi}{r^2\sin^2\theta}.
\end{aligned}$$

Following the solution of $\nabla^2\varphi = -\frac{\rho}{\varepsilon}$, we have

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}')}{r}\mathrm{d}V', \quad r = |\boldsymbol{x} - \boldsymbol{x}'|. \tag{3.2}$$

Then,

$$\nabla \times \boldsymbol{A} = \frac{\mu}{4\pi}\int_V \nabla \times \left(\frac{\boldsymbol{J}(\boldsymbol{x}')}{r}\right)\mathrm{d}V' = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}') \times \boldsymbol{r}}{r^3}\mathrm{d}V'.$$

which is the Biot-Savart law.

For a thin wire carrying current I along curve L, we have $\boldsymbol{J}\mathrm{d}V' = I\mathrm{d}\boldsymbol{l}'$, hence

$$\boldsymbol{A} = \frac{\mu}{4\pi}\int_L \frac{I\mathrm{d}\boldsymbol{l}'}{r}.$$

By the boundary condition of magnetic field, we have

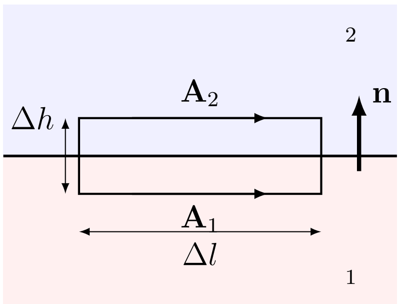
$$\boldsymbol{n}\cdot(\nabla \times \boldsymbol{A}_2 - \nabla \times \boldsymbol{A}_1) = 0,$$
$$\boldsymbol{n} \times \left(\frac{1}{\mu_2}\nabla \times \boldsymbol{A}_2 - \frac{1}{\mu_1}\nabla \times \boldsymbol{A}_1\right) = \boldsymbol{\alpha}.$$

Consider a small rectangular loop across the boundary with sides parallel and perpendicular to the boundary surface, we have

$$\oint_L \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = (A_{2t} - A_{1t})\,\Delta l, \quad \oint_L \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \to 0 \Rightarrow A_{2t} = A_{1t}.$$

Since $\nabla\cdot\boldsymbol{A} = 0$, we have $A_{2n} = A_{1n}$ and $\boldsymbol{n}\cdot(\boldsymbol{A}_2 - \boldsymbol{A}_1) = 0$. Together, we have

$$\boldsymbol{A}_2 = \boldsymbol{A}_1. \tag{3.3}$$



Proposition 3.3.

$$W = \frac{1}{2}\int_V \boldsymbol{A}\cdot\boldsymbol{J}\mathrm{d}V. \tag{3.4}$$

Proof.

$$\boldsymbol{B}\cdot\boldsymbol{H} = (\nabla \times \boldsymbol{A})\cdot\boldsymbol{H} = \nabla\cdot(\boldsymbol{A} \times \boldsymbol{H}) + \boldsymbol{A}\cdot\boldsymbol{J}.$$

$$W = \frac{1}{2}\int \boldsymbol{B}\cdot\boldsymbol{H}\mathrm{d}V = \frac{1}{2}\oint_\infty \boldsymbol{A} \times \boldsymbol{H}\mathrm{d}S + \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{J}\mathrm{d}V = \frac{1}{2}\int_V \boldsymbol{A}\cdot\boldsymbol{J}\mathrm{d}V.$$

□

Consider $\boldsymbol{J}$ in a given magnetic field with $\boldsymbol{A}_e$ and $\boldsymbol{J}_e$ as the electric field and current density, respectively. We have the total energy

$$W = \int (\boldsymbol{J} + \boldsymbol{J}_e) \cdot (\boldsymbol{A} + \boldsymbol{A}_e) \, \mathrm{d}V,$$

and the interaction energy

$$W_i = \int (\boldsymbol{J} \cdot \boldsymbol{A}_e + \boldsymbol{J}_e \cdot \boldsymbol{A}) \, \mathrm{d}V.$$

Since $\boldsymbol{A} = \frac{\mu}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V'}{r}$ and $\boldsymbol{A}_e = \frac{\mu}{4\pi} \int \frac{\boldsymbol{J}_e(\boldsymbol{x}')\mathrm{d}V'}{r}$, we have

$$W_i = \int_V \boldsymbol{J} \cdot \boldsymbol{A}_e \mathrm{d}V. \tag{3.5}$$

3.2 Uniqueness Theorem

Theorem 3.4 (Uniqueness Theorem)**.** *If there exists currents and magnetic medium such that $\boldsymbol{B} = \mu \boldsymbol{H}$ in a magnetostatic body, then the magnetic field is uniquely determined by the tangent component of $\boldsymbol{A}$ or $\boldsymbol{H}$ on the boundary.*

Proof. Suppose there are two solutions $\boldsymbol{B}'$ and $\boldsymbol{B}''$, then we have

$$\boldsymbol{B}' = \nabla \times \boldsymbol{A}', \quad \boldsymbol{B}' = \mu \boldsymbol{H}',$$

$$\boldsymbol{B}'' = \nabla \times \boldsymbol{A}'', \quad \boldsymbol{B}'' = \mu \boldsymbol{H}'', \quad \nabla \times \boldsymbol{H}' = \nabla \times \boldsymbol{H}'' = \boldsymbol{J}.$$

Construct a new magnetic field $\boldsymbol{B} = \boldsymbol{B}' - \boldsymbol{B}''$ and $\boldsymbol{H} = \boldsymbol{H}' - \boldsymbol{H}''$, then we have

$$\boldsymbol{A} = \boldsymbol{A}' - \boldsymbol{A}'', \quad \nabla \times \boldsymbol{H} = 0.$$

The total energy of this new field is

$$\begin{aligned} W &= \frac{1}{2} \int \boldsymbol{B} \cdot \boldsymbol{H} \mathrm{d}V = \frac{1}{2} \int_V (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{H} \mathrm{d}V \\ &= \frac{1}{2} \int_V \nabla \cdot (\boldsymbol{A} \times \boldsymbol{H}) \, \mathrm{d}V + \frac{1}{2} \int_V \boldsymbol{A} \cdot (\nabla \times \boldsymbol{H}) \mathrm{d}V = \frac{1}{2} \oint_\infty (\boldsymbol{A} \times \boldsymbol{H}) \mathrm{d}S \\ &= \frac{1}{2} \oint_S (\boldsymbol{n} \times (\boldsymbol{A}' - \boldsymbol{A}'')) \cdot (\boldsymbol{H}' - \boldsymbol{H}'') \, \mathrm{d}S \\ &= \frac{1}{2} \oint_S (\boldsymbol{n} \times (\boldsymbol{H}' - \boldsymbol{H}'')) \cdot (\boldsymbol{A}' - \boldsymbol{A}'') \, \mathrm{d}S = 0. \end{aligned}$$

Since $W = \frac{1}{2} \int_V \frac{1}{\mu} (\boldsymbol{B}' - \boldsymbol{B}'') \cdot (\boldsymbol{B}' - \boldsymbol{B}'') \, \mathrm{d}V$ and $\boldsymbol{n} \times \boldsymbol{A}' = \boldsymbol{n} \times \boldsymbol{A}''$ on the boundary, we have $\int \frac{1}{\mu} (\boldsymbol{B}' - \boldsymbol{B}'') \cdot (\boldsymbol{B}' - \boldsymbol{B}'') \, \mathrm{d}V = 0$, which implies $\boldsymbol{B}' = \boldsymbol{B}''$ in the region. Or similarly, by $\boldsymbol{n} \times \boldsymbol{H}' = \boldsymbol{n} \times \boldsymbol{H}''$, we have the same conclusion. □

3.3 Magnetic Scalar Potential

By Ampere's law, we have $\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{J}_f \cdot \mathrm{d}\boldsymbol{S}$, so generally, we cannot define a scalar potential for $\boldsymbol{H}$. However, in a region where all loops do not enclose any current, i.e. a simply connected region without free current distribution, we can define a scalar potential.

$$\nabla \cdot \boldsymbol{B} = 0, \quad \boldsymbol{B} = \mu(\boldsymbol{H} + \boldsymbol{M}) \Longrightarrow \nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}.$$

Let $\rho_m = -\nabla \cdot \boldsymbol{M}$, then we have $\nabla \cdot \boldsymbol{H} = \frac{\rho_M}{\mu_0}$ and $\nabla \times \boldsymbol{H} = 0$. Hence we can define the **magnetic scalar potential** φ_m such that

$$\boldsymbol{H} = -\nabla\varphi_m, \quad \nabla^2\varphi_m = -\frac{\rho_m}{\mu_0}.$$

The boundary conditions are

$$\varphi_{m2} = \varphi_{m1}(\text{ when } \boldsymbol{\alpha}_f = 0), \quad \boldsymbol{n} \times (\nabla(\varphi_{m2} - \varphi_{m1})) = -\boldsymbol{\alpha}_f.$$

For non-ferromagnetic media, we have $\mu_1 \frac{\partial \varphi_{n1}}{\partial n} = \mu_2 \frac{\partial \varphi_{n2}}{\partial n}$. For ferromagnetic media, we have $\frac{\partial \varphi_{m2}}{\partial n} - \frac{\partial \varphi_{m1}}{\partial n} = \boldsymbol{n} \cdot (\boldsymbol{M}_2 - \boldsymbol{M}_1) = -\frac{\sigma_m}{\mu_0}$, where $\sigma_m = -\mu_0 \boldsymbol{n} \cdot (\boldsymbol{M}_2 - \boldsymbol{M}_1)$ is the surface magnetic charge density.

Now we compare the electrostatics and magnetostatics:

Electrostatics	Magnetostatics
$\nabla \times \boldsymbol{E} = 0$	$\nabla \times \boldsymbol{H} = 0$
$\nabla \cdot \boldsymbol{E} = \frac{\rho_f + \rho_p}{\varepsilon_0}$	$\nabla \cdot \boldsymbol{H} = \frac{\rho_m}{\mu_0}$
$\rho_p = -\nabla \cdot \boldsymbol{P}$	$\rho_m = -\nabla \cdot \boldsymbol{M}$
$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$	$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$
$\boldsymbol{E} = -\nabla\varphi_e$	$\boldsymbol{H} = -\nabla\varphi_m$
$\nabla^2\varphi_e = -\frac{\rho_f + \rho_p}{\varepsilon_0}$	$\nabla^2\varphi_m = -\frac{\rho_m}{\mu_0}$
$\sigma_p = \boldsymbol{n} \cdot (\boldsymbol{P}_2 - \boldsymbol{P}_1)$	$\sigma_m = -\mu_0 \boldsymbol{n} \cdot (\boldsymbol{M}_2 - \boldsymbol{M}_1)$
$\boldsymbol{E}$	$\boldsymbol{H}$
$\boldsymbol{D}$	$\boldsymbol{B}$
ε_0	μ_0
$\boldsymbol{P}$	$\mu_0 \boldsymbol{H}$
$\rho_f + \rho_p$	ρ_m
σ_p	σ_m

Example 3.5. Show that the suface of a magnetic body with $\mu \to \infty$ is an equipotential surface of the magnetic scalar potential.

Proof. By boundary conditions

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0, \quad \boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0, \quad \boldsymbol{B}_2 = \mu_0 \boldsymbol{H}_2, \quad \boldsymbol{B}_1 = \mu \boldsymbol{H}_1,$$

we have

$$\mu_0 H_{2n} = \mu H_{1n}, \quad H_{2t} = H_{1t} \Longrightarrow \frac{H_{2t}}{H_{1t}} = \frac{\mu_0}{\mu} \frac{H_{1n}}{H_{2n}} \to 0.$$

Hence, on the surface, $\boldsymbol{H}_2$ is normal to the surface and the surface is an equipotential surface of φ_m. □

Example 3.6. Find the magnetic field of a homogeneous magnetic sphere with $\boldsymbol{M}_0$.

Proof. Since $\boldsymbol{M} = 0$ outside the sphere and $\boldsymbol{M} = \boldsymbol{M}_0$ inside the sphere, for both regionswe have

$$\rho_m = -\mu_0 \nabla \cdot \boldsymbol{M} = 0 \Longrightarrow \nabla^2 \varphi_1 = 0, \quad \nabla^2 \varphi_2 = 0.$$

Since $\varphi_1(\infty) = 0$ and $\varphi_2(0)$ is finite, the general solutions are

$$\varphi_1 = \sum_{n=0}^{\infty} \frac{b_n}{R^{n+1}} P_n(\cos\theta), \quad \varphi_2 = \sum_{n=0}^{\infty} a_n R^n P_n(\cos\theta).$$

On the surface, we have

$$\sum_{n=0}^{\infty} \frac{b_n}{R_0^{n+1}} P_n(\cos\theta) = \sum_{n=0}^{\infty} a_n R_0^n P_n(\cos\theta).$$

Since $\frac{\partial \varphi_2}{\partial n} - \frac{\partial \varphi_1}{\partial n} = -\boldsymbol{n} \cdot (\boldsymbol{M}_2 - \boldsymbol{M}_1) = \boldsymbol{r} \cdot \boldsymbol{M}_0$ where $\boldsymbol{r}$ is the unit vector normal to the surface, we have

$$\sum_{n=0}^{\infty} \left(\frac{(n+1)b_n}{R_0^{n+2}} + n a_n R_0^{n-1} \right) P_n(\cos\theta) = M_0 \cos\theta.$$

Hence, we have

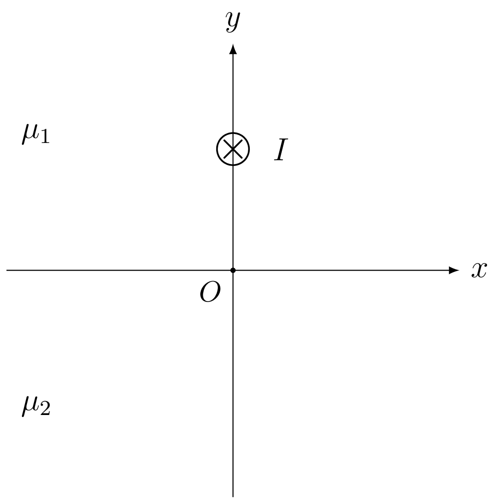
$$a_1 = \frac{M_0}{3}, \quad b_1 = \frac{M_0 R_0^3}{3}, \quad a_n = b_n = 0 \text{ for } n \neq 1,$$

and

$$\varphi_1 = \frac{M_0 R_0^3}{3R^2} \cos\theta = \frac{R_0^3}{3} \frac{\boldsymbol{M}_0 \cdot \boldsymbol{R}}{R^3}, \quad \varphi_2 = \frac{M_0}{3} R \cos\theta = \frac{1}{3} \boldsymbol{M}_0 \cdot \boldsymbol{R}.$$

$$m = \frac{4\pi R_0^3}{3} \boldsymbol{M}_0 = \boldsymbol{M}_0 V, \quad \boldsymbol{H} = -\nabla \varphi_2 = -\frac{1}{3} \boldsymbol{M}_0, \quad \boldsymbol{B} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}_0) = \frac{2}{3} \mu_0 \boldsymbol{M}_0.$$

□



3.4 Multipole Expansion

Consider the expansion of $\boldsymbol{A}(\boldsymbol{x})$ when $r = |\boldsymbol{x} - \boldsymbol{x}'| \gg 1$.

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}')}{r}\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \boldsymbol{J}(\boldsymbol{x}')\left(\frac{1}{R} - (\boldsymbol{x}'\cdot\nabla)\frac{1}{R} + \frac{1}{2}(\boldsymbol{x}'\boldsymbol{x}' : \nabla\nabla)\frac{1}{R} + \cdots\right) \\
&= \frac{\mu_0}{4\pi}\frac{1}{R}\int_V \boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V'\cdot\frac{1}{R} - \frac{\mu_0}{4\pi R^3}\int_V \boldsymbol{J}(\boldsymbol{x}')\boldsymbol{x}'\mathrm{d}V'\cdot\frac{1}{R} \\
&- \frac{\mu_0}{24\pi R^3}\int_V \boldsymbol{J}(\boldsymbol{x}')(3\boldsymbol{x}'\boldsymbol{x}' - r'^2\vec{\vec{I}})\mathrm{d}V' : \nabla\nabla\frac{1}{R} + \cdots \\
&= \boldsymbol{A}^{(0)}(\boldsymbol{x}) + \boldsymbol{A}^{(1)}(\boldsymbol{x}) + \boldsymbol{A}^{(2)}(\boldsymbol{x}) + \cdots
\end{aligned}$$

We can divide the currents into some closed loops. For each loop, we have

$$\int_V \boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V' = I\oint_L \mathrm{d}\boldsymbol{l}' = 0 \Longrightarrow \boldsymbol{A}^{(0)}(\boldsymbol{x}) = \frac{\mu_0}{4\pi R}\int_V \boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V' = 0. \tag{3.6}$$

Notice that

$$\begin{aligned}
&\oint(\boldsymbol{x}\cdot\boldsymbol{R})\,\mathrm{d}\boldsymbol{l}' + \oint(\mathrm{d}\boldsymbol{l}'\cdot\boldsymbol{x}) = \oint\mathrm{d}\left((\boldsymbol{x}'\cdot\boldsymbol{R})\cdot\boldsymbol{x}'\right) = 0 \Longrightarrow \\
&\oint(\boldsymbol{x}\cdot\boldsymbol{R})\,\mathrm{d}\boldsymbol{l}' = \frac{1}{2}\oint\left(((\boldsymbol{x}'\cdot\boldsymbol{R})\cdot\boldsymbol{x}')\,\mathrm{d}\boldsymbol{l}' - (\mathrm{d}\boldsymbol{l}'\cdot\boldsymbol{R})\,\boldsymbol{x}\right) = \frac{1}{2}\oint(\boldsymbol{x}'\times\mathrm{d}\boldsymbol{l}')\times\boldsymbol{R}.
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}^{(1)}(\boldsymbol{x}) &= -\frac{\mu_0 I}{4\pi}\oint\left(\boldsymbol{x}'\cdot\nabla\frac{1}{R}\right)\mathrm{d}\boldsymbol{l}' = \frac{\mu_0 I}{4\pi}\oint(\boldsymbol{x}'\cdot\boldsymbol{R})\frac{\mathrm{d}\boldsymbol{l}'}{R^3} \\
&= \frac{\mu_0 I}{8\pi R^3}\oint(\boldsymbol{x}'\times\mathrm{d}\boldsymbol{l}')\times\boldsymbol{R} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{R}}{R^3},
\end{aligned} \tag{3.7}$$

where $\boldsymbol{m} = \frac{I}{2}\oint\boldsymbol{x}'\times\mathrm{d}\boldsymbol{l}'$ is the **magnetic moment** of the current loop. For body current distribution, we have

$$\boldsymbol{m} = \frac{1}{2}\int_V \boldsymbol{x}'\times\boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V'.$$

For a small current loop with area ΔS, we have

$$\Delta S = \frac{1}{2}\oint\boldsymbol{x}'\times\mathrm{d}\boldsymbol{l}', \quad \boldsymbol{m} = I\Delta S.$$

Then we can compute the magnetic field of a magnetic dipole. Since

$$\begin{aligned}
\nabla\left(\boldsymbol{m}\cdot\frac{\boldsymbol{R}}{R^3}\right) &= (\boldsymbol{m}\cdot\nabla)\frac{\boldsymbol{R}}{R^3} + \boldsymbol{m}\times\left(\nabla\times\frac{\boldsymbol{R}}{R^3}\right) + \frac{\boldsymbol{R}}{R^3}\times(\nabla\times\boldsymbol{m}) + \left(\frac{\boldsymbol{R}}{R^3}\cdot\nabla\right)\boldsymbol{m} \\
&= (\boldsymbol{m}\cdot\nabla)\frac{\boldsymbol{R}}{R^3},
\end{aligned}$$

we have

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{x}) &= \nabla\times\boldsymbol{A}^{(1)}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{\boldsymbol{m}\times\boldsymbol{R}}{R^3}\right) = \frac{\mu_0}{4\pi}\left(\boldsymbol{m}\left(\nabla\cdot\frac{\boldsymbol{R}}{R^3}\right) - (\boldsymbol{m}\cdot\nabla)\frac{\boldsymbol{R}}{R^3}\right) \\
&= -\mu_0\nabla\left(\frac{\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3}\right) = -\mu_0\nabla\varphi_m^{(1)}, \quad \varphi_m^{(1)} = \frac{\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3}.
\end{aligned} \tag{3.8}$$

Any current loop can be viewed as a combination of many small current loops on S, hence we have $\boldsymbol{m} = I \int_S \mathrm{d}\boldsymbol{S}$.

Since the interaction energy is $W = \int_V \boldsymbol{J}(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x})\mathrm{d}V$, we have

$$W = I \oint_L \boldsymbol{A}_e \cdot \mathrm{d}\boldsymbol{l} = I \int_S \boldsymbol{B}_e \cdot \mathrm{d}\boldsymbol{S} = I\Phi_e \tag{3.9}$$

By the expansion of $\boldsymbol{B}_e$, we have

$$W \approx I\boldsymbol{B}_e(0) \cdot \int_S \mathrm{d}\boldsymbol{S} = \boldsymbol{m} \cdot \boldsymbol{B}_e(0) \tag{3.10}$$

The potential for magnetic dipole is $U = -\boldsymbol{m} \cdot \boldsymbol{B}_e(0)$, we have the force

$$F = -\nabla U = \boldsymbol{m} \cdot \nabla \boldsymbol{B}_e(0), \tag{3.11}$$

the torque is

$$L = -\frac{\partial}{\partial\theta}U = \frac{\partial}{\partial\theta} mB_e \cos\theta = -mB_e \sin\theta, \quad \boldsymbol{L} = \boldsymbol{m} \times \boldsymbol{B}_e. \tag{3.12}$$

3.5 Aharonov-Bohm Effect

Proposition 3.7 (Aharonov-Bohm Effect)**.**

Outside the loop, $\boldsymbol{B} = 0$, but $\boldsymbol{A} \neq 0$, since

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi.$$

This effect shows that $\boldsymbol{A}$ has observable physical effects. The wave function of free electron is

$$\psi_0(\boldsymbol{x}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}}.$$

When $\boldsymbol{A} = 0$, we have

$$\Delta\Phi_0 = \frac{1}{\hbar}\left(\int_{C_2} \boldsymbol{p}_2 \cdot \mathrm{d}\boldsymbol{l} - \int_{C_1} \boldsymbol{p}_1 \cdot \mathrm{d}\boldsymbol{l}\right) = \frac{1}{\hbar}p\Delta l = \frac{1}{\hbar}pd\sin\theta \approx \frac{1}{\hbar}pd\frac{y}{L}.$$

When $\boldsymbol{A} \neq 0$, use the regular momentum and we have

$$\psi(\boldsymbol{x}) = e^{\frac{i}{\hbar}\int \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{l}}, \quad \boldsymbol{P} = \boldsymbol{p} - e\boldsymbol{A} = m\boldsymbol{v} - e\boldsymbol{A}.$$

$$\begin{aligned}\Delta\Phi &= \frac{1}{\hbar}\left(\int_{C_2} \boldsymbol{P}_2 \cdot \mathrm{d}\boldsymbol{l} - \int_{C_1} \boldsymbol{P}_1 \cdot \mathrm{d}\boldsymbol{l}\right) = \Delta\Phi_0 - \frac{e}{\hbar}\left(\int_{C_2} \boldsymbol{A}_2 \cdot \mathrm{d}\boldsymbol{l} - \int_{C_1} \boldsymbol{A}_1 \cdot \mathrm{d}\boldsymbol{l}\right) \\ &= \Delta\Phi_0 - \frac{e}{\hbar}\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Delta\Phi_0 - \frac{e}{\hbar}\Phi.\end{aligned}$$

Since $\Delta\Phi - \Delta\Phi_0$, the image will move y_0.

$$\Delta\Phi_0 = \frac{e}{\hbar}\Phi \Longrightarrow y_0 = \frac{e\Phi L}{mvd}.$$

To describe the magnetic field completely and properly, we have $e^{i\frac{e}{\hbar}\oint_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}$. When C shrinks to a infinitely small path, then $\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{B} \cdot \Delta S$.

3.6 Electromagnetic Properties of Superconductors

T_c is the transition temperature. When $T < T_c$, the resistance rate turns to 0.

$$B_c = B_{c_0}\left(1 - \frac{T^2}{T_c^2}\right).$$

$$I_c = I_{c_0}\left(1 - \frac{T^2}{T_c^2}\right).$$

Proposition 3.8 (Meissner Effect)**.** *For a superconductor, the inner magnetic fieledd is zero.*

There are normal electrons n_n and superconductor electrons n_s, then $n = n_n + n_s$. When 0 K, $n_s = N$ and $n_n = 0$. When T increases, n_n increases and n_s decreases. When $T = T_c$, $n_n = N$ and $n_s = 0$. Normal electrons follow the Ohm Law $\boldsymbol{J}_n = \sigma \boldsymbol{E}$. The superconductor electrons follow

$$m\frac{\partial \boldsymbol{v}}{\partial t} = -e\boldsymbol{E} \Rightarrow \frac{\partial \boldsymbol{J}_s}{\partial t} = -\frac{\partial n_s e\boldsymbol{v}}{\partial t} = \alpha \boldsymbol{E}, \quad \alpha = \frac{n_s e^2}{m}.$$

For steady, $\frac{\partial \boldsymbol{J}_s}{\partial t} = 0$, we have $\boldsymbol{E} = 0$ and $\boldsymbol{J}_n = 0$.

$$\frac{\partial}{\partial t}\nabla \times \boldsymbol{J}_s = \alpha \nabla \times \boldsymbol{E} = -\alpha\frac{\partial \boldsymbol{B}}{\partial t}.$$

$$\nabla \times \boldsymbol{J}_s = -\alpha \boldsymbol{B} + f(\boldsymbol{x})$$

By $\nabla \times \boldsymbol{J}_s = -\frac{n_s e^2}{m}\boldsymbol{B}$ and $\nabla \cdot \boldsymbol{J}_s = 0$, we have

$$\nabla^2 \boldsymbol{J}_s = \frac{1}{\lambda_L^2}\boldsymbol{J}_s. \tag{3.13}$$

Hence the current mainly distributes on the surface with thickness λ_L.

$$\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_M, \quad \nabla \times \boldsymbol{M} = \boldsymbol{J}_s, \quad \boldsymbol{n} \times \boldsymbol{M} = -\alpha_s.$$

Since $\boldsymbol{B} = 0$ inside the superconductor and $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$, we have $\boldsymbol{H} = -\boldsymbol{M}$ and $\chi_m = -1$, $\mu = \mu_0(1 + \chi_m) = 0$.

Now we introduce the quantization of magnetic flux. Consider a superconducting ring. First, we show that the magnetic flux is a constant.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0.$$

Since the wave function is single-valued, the phase change along the loop is $2n\pi$, i.e.

$$\begin{aligned}\Delta\Phi &= \frac{1}{\hbar}\oint_C \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{l} = \frac{1}{\hbar}\oint_C (2m\boldsymbol{v} - 2e\boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{l} = \frac{1}{\hbar}\oint_C \left(-\frac{2m}{n_s e}\boldsymbol{J}_s - 2e\boldsymbol{A}\right) \cdot \mathrm{d}\boldsymbol{l} \\ &= -\frac{2e}{\hbar}\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = -\frac{2e}{\hbar}\Phi = 2n\pi \Longrightarrow \Phi = n\frac{h}{2e} = n\Phi_0,\end{aligned}$$

where $\Phi_0 = \frac{h}{2e}$ is the **flux quantum**. Generally, the quantization of magnetic flux exists in any doubly connected superconductor.

In BCS theory by Bardeen, Cooper and Schrieffer, the electrons form Cooper pairs with binding energy Δ. When $kT < \Delta$, the Cooper pairs does not break and when $kT > \Delta$, the Cooper pairs break, i.e. $T_c = \frac{\Delta}{k}$.

Pippard

4 Spread of Electromagnetic Waves

4.1 Plane Electromagnetic Waves

In free space without currents and charges, we have Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{D} = 0.$$

Hence we have

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{H}) = -\mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}.$$

Since $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E}$, we have the wave equation

$$\nabla^2 \boldsymbol{E} - \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0, \quad \nabla^2 \boldsymbol{B} - \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0. \tag{4.1}$$

Denote $c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$, then c_0 is the speed of wave in vacuum and we have

$$\nabla^2 \boldsymbol{E} - \frac{1}{c_0^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0, \quad \nabla^2 \boldsymbol{B} - \frac{1}{c_0^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0. \tag{4.2}$$

There is dispersion for medium case. For linear medium, we have

$$\boldsymbol{D}(\omega) = \varepsilon(\omega)\boldsymbol{E}(\omega), \quad \boldsymbol{B}(\omega) = \mu(\omega)\boldsymbol{H}(\omega).$$

Definition 4.1. Electromagnetic wave with one frequency is called **monochromatic wave** or **harmonic wave**.

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}, \quad \boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x})e^{-i\omega t}. \tag{4.3}$$

Generally, we can use Fourier transform to represent any wave as a combination of monochromatic waves with different frequencies, hence we only consider monochromatic waves. For Maxwell equations, we have

$$\nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = -i\omega\varepsilon\boldsymbol{E}, \quad \nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0. \tag{4.4}$$

Actually, $\nabla \cdot (\nabla \times \boldsymbol{E}) = 0 \Longrightarrow \nabla \cdot \boldsymbol{H} = 0$ and $\nabla \cdot (\nabla \times \boldsymbol{H}) = 0 \Longrightarrow \nabla \cdot \boldsymbol{E} = 0$. Hence only the first two equations are independent. And we have the **Helmholtz equation**

$$\nabla \times (\nabla \times \boldsymbol{E}) = i\omega\mu(\nabla \times \boldsymbol{H}) = \omega^2\mu\varepsilon\boldsymbol{E} \Longrightarrow \nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = 0, \quad k = \omega\sqrt{\mu\varepsilon}. \tag{4.5}$$

Then we have

$$\boldsymbol{B} = -\frac{i}{\omega}\nabla \times \boldsymbol{E} = -\frac{i}{k}\sqrt{\mu\omega}\nabla \times \boldsymbol{E}. \tag{4.6}$$

Every solution of Helmholtz equation is called a **mode** of the electromagnetic wave.

Now we discuss the simplest mode, the **plane wave**. Then we have

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 t}\boldsymbol{E}\boldsymbol{x} + k^2\boldsymbol{E}(\boldsymbol{x}) = 0 \Longrightarrow \boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}.$$

When $t = 0$, the phase factor is $\cos kx$ and $x = 0$ is a wave crest. When t, the phase factor is $\cos(kx - \omega t)$, hence the wave crest moves to $x = \frac{\omega}{k}t$. This describes a wave moving along the x-axis.

The phase speed and group speed are

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{\sqrt{\mu\varepsilon}}. \tag{4.7}$$

$$\nabla \cdot \boldsymbol{E} = i\boldsymbol{k} \cdot \boldsymbol{E} = 0 \Longrightarrow \boldsymbol{k} \cdot \boldsymbol{E} = 0.$$

The direction of $\boldsymbol{E}$ is the polarization direction, hence every $\boldsymbol{k}$ corresponds to two independent polarization directions.

$$\nabla \times \boldsymbol{E} = \left(\nabla e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\right) \times \boldsymbol{E}_0 = i\boldsymbol{k} \times \boldsymbol{E} \Longrightarrow \boldsymbol{B} = \sqrt{\mu\varepsilon}\,\boldsymbol{e}_k \times \boldsymbol{E}.$$

Hence, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ are mutually perpendicular and electromagnetic wave is a transverse wave. The relation between the amplitudes is

$$\left|\frac{\boldsymbol{E}}{\boldsymbol{B}}\right| = \frac{1}{\sqrt{\varepsilon\mu}} = v, \quad \text{in vacuum } \left|\frac{\boldsymbol{E}}{\boldsymbol{B}}\right| = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = c. \tag{4.8}$$

Next, we compute the energy density and energy flux density of electromagnetic waves. The energy density in linear homogeneous medium is

$$\omega = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}) = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right).$$

For plane wave, we have

$$\varepsilon E^2 = \frac{1}{\mu}B^2 \Longrightarrow \omega = \varepsilon E^2 = \frac{1}{\mu}B^2.$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \sqrt{\frac{\varepsilon}{\mu}}\boldsymbol{E} \times (\boldsymbol{e}_k \times \boldsymbol{E}) = \sqrt{\frac{\varepsilon}{\mu}}E^2\boldsymbol{e}_k = v\omega\boldsymbol{e}_k, \quad v = v_p. \tag{4.9}$$

Since we only consider the real part of energy, we have

$$\omega = \varepsilon E_0^2 \cos^2(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \Longrightarrow \overline{\omega} = \frac{1}{2}\varepsilon E_0^2.$$

In general, for complex numbers $f(t) = f_0 e^{-i\omega t}$ and $g(t) = g_0 e^{-i\omega t + i\phi}$, the time average of their product is

$$\overline{fg} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \mathrm{d}t f_0 \cos(\omega t) g_0 \cos(\omega t - \phi) = \frac{1}{2}f_0 g_0 \cos\phi = \frac{1}{2}\Re(f_0^* g_0).$$

$$\overline{\boldsymbol{S}} = \frac{1}{2}\Re(\boldsymbol{E}^* \times \boldsymbol{H}) = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}E_0^2\boldsymbol{e}_k. \tag{4.10}$$

4.2 Reflection and Refraction of Electromagnetic Waves on the Boundary

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{E}' = \boldsymbol{E}'_0 e^{i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{E}'' = \boldsymbol{E}''_0 e^{i(\boldsymbol{k}''\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k} = (k_x, 0, k_z).$$

If $\varepsilon_1 > \varepsilon_2$, then $n_{21} < 1$ and $\theta'' > \theta$. Let $\sin\theta'' = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta = \frac{\sin\theta}{n_{21}} = 1$, then $\theta'' = \frac{\pi}{2}$.

$$k''_x = k_x = k\sin\theta, \quad k'' = k\frac{v_1}{v_2} = kn_{21} \Longrightarrow k''_x > k''.$$

Hence

$$k''_z = \sqrt{k''^2 - k''^2_x} = ik\sqrt{\sin^2\theta - n_{21}^2} = i\kappa, \quad \kappa = k\sqrt{\sin^2\theta - n_{21}^2}.$$

Then the transmitted wave is

$$\boldsymbol{E}'' = \boldsymbol{E}''_0 e^{-\kappa z} e^{i(k''_x - \omega t)}. \tag{4.11}$$

The average energy flux density is

$$\overline{S''} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}^* \times \boldsymbol{H}\right) = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}''^* \times \left(\sqrt{\frac{\varepsilon_2}{\mu_2}}\frac{\boldsymbol{k}''}{k''}\right)\right) = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}^* \times \left(\sqrt{\frac{\varepsilon_2}{\mu_2}}\frac{\boldsymbol{k}''}{k''}\right)\right) = \tag{4.12}$$

Hence $\overline{S''}_z = 0$ and only $\overline{S''}_x$ exists.

$$\frac{\boldsymbol{E}'}{\boldsymbol{E}} = \frac{\sqrt{\varepsilon_1}\cos\theta - \sqrt{\varepsilon_2}\cos\theta''}{\sqrt{\varepsilon_1}\cos\theta + \sqrt{\varepsilon_2}\cos\theta''} = e^{-2i\phi}, \quad \tan\phi = \frac{\sqrt{\sin^2\theta - n_{21}^2}}{\cos\theta}. \tag{4.13}$$

4.3 Spread of Electromagnetic Waves in Conductors

In vacuum or ideal dielectrics, there is no energy loss. In conductors, there is energy loss.

$$\frac{\partial\rho_f}{\partial t} = -\nabla\cdot\boldsymbol{J} = -\frac{\sigma}{\varepsilon}\rho_f \Longrightarrow \rho_f(t) = \rho_{f0}e^{-\frac{\sigma}{\varepsilon}t}.$$

The eigen time τ of decay is the time when $\rho_f(\tau) = \frac{1}{e}\rho_{f0}$, i.e.

$$\tau = \frac{\varepsilon}{\sigma}.$$

Hence, we have the good conductor condition:

$$\omega \ll \tau^{-1} = \frac{\sigma}{\varepsilon}, \quad \frac{\sigma}{\varepsilon\omega} \gg 1 \Longrightarrow \rho_f \approx 0. \tag{4.14}$$

For general metal conductors, $\sigma \sim 10^7$ s, hence if ω is not very high, the good conductor condition holds.

Since $\rho = 0$ and $\boldsymbol{J} = \sigma\boldsymbol{E}$, we have the Maxwell equations

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t},\\
\nabla\times\boldsymbol{H} &= \frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{J},\\
\nabla\cdot\boldsymbol{B} &= 0,\\
\nabla\cdot\boldsymbol{D} &= 0.
\end{aligned}$$

Substituting $\boldsymbol{B} = \mu \boldsymbol{H}$, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, we have

$$\begin{aligned} \nabla \times \boldsymbol{E} &= i\omega\mu \boldsymbol{H}, \\ \nabla \times \boldsymbol{H} &= -i\omega\varepsilon \boldsymbol{E} + \sigma \boldsymbol{E} = -i\omega\varepsilon' \boldsymbol{E}, \quad \varepsilon' = \varepsilon + \frac{i\sigma}{\omega}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \cdot \boldsymbol{D} &= 0. \end{aligned} \tag{4.15}$$

The two items in the right hand side of second equation represent the displacement current and conduction current respectively. The energy density of conduction current is $\frac{1}{2}\mathrm{Re}\left(\boldsymbol{J}^* \cdot \boldsymbol{E}\right) = \frac{1}{2}\sigma E_0^2$. The energy density of displacement current is 0.

Then we have the Helmholtz equation

$$\begin{aligned} &\nabla^2 \boldsymbol{E} - \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \Longrightarrow \nabla^2 \boldsymbol{E} + \omega^2 \left(\varepsilon\mu + i\mu\frac{\sigma}{\omega}\right) \boldsymbol{E} = 0, \\ &\nabla^2 \boldsymbol{B} - \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0. \end{aligned}$$

Hence we have the modified Helmholtz equation

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0, \quad k = \omega\sqrt{\mu\varepsilon'}. \tag{4.16}$$

The plane wave solution is $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, where $\boldsymbol{k} = \boldsymbol{\beta} + i\boldsymbol{\alpha}$ is a complex number in general. Then we have

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 e^{-\boldsymbol{\alpha}\cdot\boldsymbol{x}} e^{i(\boldsymbol{\beta}\cdot\boldsymbol{x} - \omega t)}. \tag{4.17}$$

$\boldsymbol{\alpha}$ is the decay coefficient, $\boldsymbol{\beta}$ is the phase coefficient. The relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$k^2 = \beta^2 - \alpha^2 + 2i\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \omega^2\mu\varepsilon + i\omega\mu\sigma \Longrightarrow \beta^2 - \alpha^2 = \omega^2\mu\varepsilon, \quad \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \frac{\omega\mu\sigma}{2}. \tag{4.18}$$

Consider the perpendicular incidence, i.e. $\boldsymbol{\alpha} = \boldsymbol{e}_z \alpha$, $\boldsymbol{\beta} = \boldsymbol{e}_z \beta$. Then we have

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}, \quad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}. \tag{4.19}$$

Since $\frac{\omega}{\varepsilon\omega} \gg 1$, we have

$$\beta \approx \alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}}. \tag{4.20}$$

Denote the **skin depth** $\delta = \frac{1}{\alpha}$, then we have

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}. \tag{4.21}$$

Proposition 4.2 (Skin Effect)**.** *For high frequency electromagnetic wave, electromagnetic field and its high-frequency current mainly distribute in the surface layer of conductor with thickness δ.*

$$\begin{aligned} &\boldsymbol{J}_d \propto \frac{\partial \boldsymbol{D}}{\partial t} \propto -i\omega\varepsilon \boldsymbol{E}, \quad \boldsymbol{J}_f = \sigma \boldsymbol{E} \Longrightarrow \left|\frac{\boldsymbol{J}_f}{\boldsymbol{J}_d}\right| = \frac{\sigma}{\omega\varepsilon} \gg 1, \quad \lambda = \frac{2\pi}{\beta} \approx \frac{1}{\beta} = \delta. \\ &\boldsymbol{H} = \frac{1}{\omega\mu}\boldsymbol{k} \times \boldsymbol{E} = \frac{1}{\omega}(\beta + i\alpha)\boldsymbol{e}_n \times \boldsymbol{E} \approx \sqrt{\frac{\sigma}{\omega\mu}} e^{i\frac{\pi}{4}} \boldsymbol{e}_n \times \boldsymbol{E} \Rightarrow \sqrt{\frac{\mu}{\varepsilon}}\left|\frac{\boldsymbol{H}}{\boldsymbol{E}}\right| = \sqrt{\frac{\mu\sigma}{\omega\varepsilon}} \gg 1. \end{aligned} \tag{4.22}$$

For dielectrics or high frequency case, $\frac{\sigma}{\varepsilon\omega} \ll 1$, $\sigma \to 0$ or $\omega \to \infty$, we have

$$\beta \approx \omega\sqrt{\frac{\varepsilon\mu}{2}} \left(1 + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 + 1\right)^{\frac{1}{2}} \approx \omega\sqrt{\varepsilon\mu}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right) \approx \omega\sqrt{\varepsilon\omega}.$$

4.4 Waveguide

Consider a rectangular waveguide with inner surface $x = 0$ and a, $y = 0$ and b. The electromagnetic wave satisfies

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E}|_{\text{boundary}} = 0, \quad k = \omega\sqrt{\mu\varepsilon}.$$

Suppose the wave spreads along the z-axis, then we have

$$\boldsymbol{E}(x,y,z) = \boldsymbol{E}(x,y)e^{ik_z z} \Longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\boldsymbol{E}(x,y) + \left(k^2 - k_z^2\right)\boldsymbol{E}(x,y) = 0.$$

Assume $u(x,y) = X(x)Y(y)$ a component of $\boldsymbol{E}(x,y)$, then we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) + k_x^2 X(x) = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y) + k_y^2 Y(y) = 0, \quad k_x^2 + k_y^2 + k_z^2 = k^2.$$

With boundary conditions, we have

$$E_x = A_1 \cos k_x x \sin k_y y e^{ik_z z}, \quad E_y = A_2 \sin k_x x \cos k_y y e^{ik_z z}, \quad E_z = A_3 \sin k_x x \sin k_y y e^{ik_z z}, \tag{4.23}$$

With boundary conditions on $x = a$ and $y = b$, we have

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad m,n = 0,1,2,\cdots \text{ are the number of half waves along } x \text{ and } y,$$

and considering $\nabla \cdot \boldsymbol{E} = 0$, we have

$$k_x A_1 + k_y A_2 - ik_z A_3 = 0.$$

Only two of A_1, A_2, A_3 are independent, hence for each pair of (m,n), there are two independent modes.

After solving $\boldsymbol{E}$, we have

$$\boldsymbol{H} = -\frac{i}{\omega\mu}\nabla \times \boldsymbol{E}.$$

Since $\boldsymbol{E}$ and $\boldsymbol{H}$ can not both be transverse waves, we often choose either $E_z = 0$ or $H_z = 0$ mode.

Definition 4.3. If $E_z = 0$, it is called **transverse electric wave** (TE wave). If $H_z = 0$, then it is **transverse magnetic wave** (TM wave). Given (m,n), they are denoted as TE_{mn} wave and TM_{mn} wave respectively.

If $k < \sqrt{k_x^2 + k_y^2}$, then k_z is imaginary and $e^{ik_z z}$ is a decay factor.

Proposition 4.4. *The lowest frequency for wave to spread in waveguide is the* ***cut-off frequency***

$$\omega_{c,mn} = \frac{\pi}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \tag{4.24}$$

If $a > b$, then TE_{10} mode has the lowest cut-off frequency

$$\omega_{c,10} = \frac{\pi}{a\sqrt{\mu\varepsilon}}, \quad \lambda_{c,10} = 2a.$$

The main mode is TE_{10} mode. TE_{10} mode has the lowest cut-off frequency, hence for some frequency, we can choose appropriate size such that only TE_{10} mode exists.

For $m = 1, n = 0$, we have $k_x = \frac{\pi}{a}$, $k_y = 0$, $A_3 = 0$. Since $k_x A_1 + k_y A_2 - ik_z A_3 = 0$, we have $A_1 = 0$. Suppose

$$A_2 = \frac{i\omega\mu a}{\pi} H_0,$$

then we have

$$E_x = E_z = H_y = 0, \quad H_x = -\frac{ik_z a}{\pi} H_0 \sin\frac{\pi x}{a}, \quad H_z = H_0 \cos\frac{\pi x}{a}, \quad E_y = \frac{i\omega\mu a}{\pi} H_0 \cos\frac{\pi x}{a}. \tag{4.25}$$

The current on the surface is given by

$$\boldsymbol{e}_n \times \boldsymbol{H} = \boldsymbol{\alpha}.$$

The phase velocity and group velocity are

$$v_p = \frac{\omega}{k_z} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1 - \left(\frac{\omega_{c,mn}}{\omega}\right)^2}} > \frac{1}{\sqrt{\mu\varepsilon}}, \quad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_z} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{1 - \left(\frac{\omega_{c,mn}}{\omega}\right)^2} < \frac{1}{\sqrt{\mu\varepsilon}}. \tag{4.26}$$

When $\omega \to \omega_{c,mn}$, $v_p \to \infty$ and $v_g \to 0$. When $\omega \to \infty$, $v_p \to \frac{1}{\sqrt{\mu\varepsilon}}$ and $v_g \to \frac{1}{\sqrt{\mu\varepsilon}}$.

5 Radiation of Electromagnetic Waves

5.1 Vector and Scalar Potentials of Electromagnetic Fields

From Maxwell equations, we have

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0 \Longrightarrow \boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}.$$

$\boldsymbol{A}$ and φ are not unique. Substituting

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\psi, \quad \varphi' = \varphi - \frac{\partial \psi}{\partial t},$$

we still have

$$\nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} = \boldsymbol{B}, \quad -\nabla\varphi' - \frac{\partial \boldsymbol{A}'}{\partial t} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E}.$$

Definition 5.1. A **gauge** is a choice of $\boldsymbol{A}$ and φ. A **gauge field** is a field that remains the same under a gauge transformation.

There are two commonly used gauges:

1. **Coulomb gauge**: $\nabla \cdot \boldsymbol{A} = 0$. Then $-\frac{\partial \boldsymbol{A}}{\partial t}$ is the transverse part of $\boldsymbol{E}$ and $-\nabla\varphi$ is the longitudinal part of $\boldsymbol{E}$.

2. **Lorentz gauge**: $\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$.

Proposition 5.2 (d'Alembert Equation)**.**

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}. \tag{5.1}$$

Proof. From Maxwell equations, we have

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0\boldsymbol{J} - \mu_0\varepsilon_0\frac{\partial}{\partial t}\nabla\varphi - \mu_0\varepsilon_0\frac{\partial^2 \boldsymbol{A}}{\partial t^2}, \quad \nabla^2\varphi + \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -\frac{\rho}{\varepsilon}$$

$$\Longrightarrow \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\boldsymbol{J}.$$

If we choose Coulomb gauge, then

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t} = -\mu_0\boldsymbol{J}, \quad \nabla^2\varphi = -\frac{\rho}{\varepsilon}, \quad \nabla \cdot \boldsymbol{A} = 0.$$

If we choose Lorentz gauge, then $\boldsymbol{A}$ and φ has the same form:

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0\boldsymbol{J}, \quad \nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}, \quad \nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0.$$

□

Example 5.3. Find the potential of a plane wave.

Proof.

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} = 0, \quad \nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} = 0, \quad \nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0.$$

The solution is

$$\boldsymbol{A} = \boldsymbol{A}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \varphi = \varphi_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \Longrightarrow i\boldsymbol{k}\cdot\boldsymbol{A}_0 - \frac{i\omega}{c^2}\varphi_0 = 0 \Longrightarrow \varphi_0 = \frac{c^2}{\omega}\boldsymbol{k}\cdot\boldsymbol{A}_0.$$

Hence, given $\boldsymbol{A}_0$, we can determine the plane wave.

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = i\boldsymbol{k}\times\boldsymbol{A},$$

$$\begin{aligned}\boldsymbol{E} &= -\nabla\varphi - \frac{\partial\boldsymbol{A}}{\partial t} = -i\boldsymbol{k}\varphi + i\omega\boldsymbol{A} = -\frac{ic^2}{\omega}\left(\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{A}\right) - k^2\boldsymbol{A}\right)\\ &= -\frac{ic^2}{\omega^2}\boldsymbol{k}\times\left(\boldsymbol{k}\times\boldsymbol{A}\right) = -\frac{c^2}{\omega}\boldsymbol{k}\times\boldsymbol{B} = c\boldsymbol{e}_k\times\boldsymbol{B}.\end{aligned}$$

If we add longitudinal component $\alpha\boldsymbol{k}$ to $\boldsymbol{A}_0$, the results do not change. Hence, even in Lorentz gauge, $\boldsymbol{A}$ is still not unique.

For simplicity, we can choose $\boldsymbol{k}\cdot\boldsymbol{A}_0 = 0 \Longrightarrow \varphi_0 = 0$ and

$$\boldsymbol{E} = i\omega\boldsymbol{A}, \quad \boldsymbol{B} = i\boldsymbol{k}\times\boldsymbol{A}.$$

If we choose Coulomb gauge, then

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t} = 0, \quad \nabla^2\varphi = 0.$$

If there is no charge, then $\varphi = 0$ and the results are the same as above. □

5.2 Retarded Potentials

We now solve the d'Alembert equation. Suppose there is a changing charge $Q(t)$ at the origin, then we have

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{1}{\varepsilon_0}Q(t)\delta(\boldsymbol{x}).$$

By the spherical symmetry, we have

$$\varphi(\boldsymbol{x},t) = \varphi(r,t), \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{1}{\varepsilon_0}Q(t)\delta(\boldsymbol{x}).$$

Since φ decreases as r increases, we denote

$$\varphi(r,t) = \frac{u(r,t)}{r} \Longrightarrow \frac{\partial^2 u}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad r \neq 0.$$

The one-dimensional wave equation has the general solution

$$u(r,t) = f(t-\frac{r}{c}) + g(t+\frac{r}{c}).$$

Since there is only outgoing wave and inspired by $\varphi = \frac{Q}{4\pi\varepsilon_0 r}$, we show that

$$\varphi(r,t) = \frac{Q\left(t-\frac{r}{c}\right)}{4\pi\varepsilon_0 r}$$

is the solution and it suffices to check $r = 0$.

$$\int_0^\eta 4\pi r^2 \mathrm{d}r \left(\nabla^2 - \frac{1}{c^2}\right) \frac{Q\left(t - \frac{r}{c}\right)}{4\pi\varepsilon_0 r} = \frac{Q(t)}{4\pi\varepsilon_0} \int_V \mathrm{d}V \nabla^2 \frac{1}{r} = -\frac{Q(t)}{\varepsilon_0}.$$

For general charge distribution $\rho(\boldsymbol{x}, t)$, we have

$$\varphi(\boldsymbol{x}, t) = \int_V \frac{\rho\left(\boldsymbol{x}', t - \frac{r}{c}\right)}{4\pi\varepsilon_0 r} \mathrm{d}V', \quad \boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}\left(\boldsymbol{x}', t - \frac{r}{c}\right)}{r} \mathrm{d}V'. \tag{5.2}$$

Suppose the distance between the field point and M_1, M_2 is r_1, r_2 respectively. Then the charge at M_1 affects $\varphi(\boldsymbol{x}, t)$ at time $t - \frac{r_1}{c}$, and the charge at M_2 affects $\varphi(\boldsymbol{x}, t)$ at time $t - \frac{r_2}{c}$. Hence, the effect of charge distribution at time t' affects the field at time $t = t' + \frac{r}{c}$, where $\frac{r}{c}$ is the time for electromagnetic wave to spread.

Definition 5.4. The potentials given by Equations 5.2 are called **retarded potentials**.

5.3 Taylor Formula of Radiation

$\boldsymbol{J}$ is a alternating current with frequency ω, i.e. $\boldsymbol{J}(\boldsymbol{x}', t) = \boldsymbol{J}(\boldsymbol{x}')e^{-i\omega t}$. Since $\rho(\boldsymbol{x}', t) = \rho(\boldsymbol{x}')e^{-i\omega t}$,

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \Longrightarrow i\omega\rho = \nabla \cdot \boldsymbol{J}.$$

Denote $k = \frac{\omega}{c}$, then we have

$$\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{A}(\boldsymbol{x})e^{-i\omega t}, \quad \boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}(\boldsymbol{x}')e^{ikr}}{r} \mathrm{d}V'. \tag{5.3}$$

According to the relation between r and wave length λ, we have three regions:

1. Near region: $r \ll \lambda$.
2. Intermediate region: $r \sim \lambda$.
3. Far region: $r \gg \lambda$.

For near region $kr \ll 1$, we have the retarded factor $e^{ikr} \approx 1$. Next we study the far region.

$$\frac{e^{ikr}}{r} = \frac{e^{ikR}}{R} - \boldsymbol{x}' \cdot \nabla \left(\frac{e^{ikr}}{r}\right) + \frac{1}{2!} \left(\boldsymbol{x}' \cdot \nabla\right)^2 \frac{e^{ikR}}{R} + \cdots, \quad R = |\boldsymbol{x}|.$$

$$\begin{aligned} |\boldsymbol{A}^{(1)}(\boldsymbol{x})| &= \frac{\mu_0}{4\pi} \left| \int_V \boldsymbol{J}(\boldsymbol{x}')\boldsymbol{x}' \mathrm{d}V' \cdot \left(-\frac{\boldsymbol{R}}{R^2} + ik\frac{\boldsymbol{R}}{R}\right) \frac{e^{ikR}}{R} \right| \\ &\leqslant \frac{\mu_0}{4\pi} \left| l' \int_V \boldsymbol{J}(\boldsymbol{x}') \mathrm{d}V' \left(-\frac{1}{R} + ik\right) \frac{e^{ikR}}{R} \right| \end{aligned}$$

where l' is the size of the source region. Since

$$\left| \frac{\boldsymbol{A}^{(1)}(\boldsymbol{x})}{\boldsymbol{A}^{(0)}(\boldsymbol{x})} \right| \leqslant l' \left| -\frac{1}{R} + ik \right| \leqslant \frac{l'}{R} + kl' \ll 1,$$

the progression converges fast if $kl' \ll 1$. Hence we only consider the first two terms.

5.4 Radiation from an Electric Dipole

$$\boldsymbol{A}^{(0)}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\int_V \boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V', \quad \int_V \boldsymbol{J}(\boldsymbol{x}')\mathrm{d}V' = \sum q_i \boldsymbol{v}_i = \frac{\mathrm{d}}{\mathrm{d}t}\sum q_i \boldsymbol{x}_i = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}.$$

$$\Longrightarrow \boldsymbol{A}^{(0)}(\boldsymbol{x}) = \frac{\mu_0 e^{ikR}}{4\pi R}\dot{\boldsymbol{p}}, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}^{(0)} = \frac{i\mu_0 k}{4\pi R}e^{ikR}\boldsymbol{e}_r\times\dot{\boldsymbol{p}}. \tag{5.4}$$

Since we only consider $\frac{1}{R}$ term, ∇ only acts on e^{ikR}, instead of $\frac{1}{R}$. Hence, we have the following substitutions:

$$\nabla \longrightarrow ik\boldsymbol{e}_r, \quad \frac{\partial}{\partial t}\longrightarrow -i\omega.$$

$$\boldsymbol{B} = ik\boldsymbol{e}_r\times\boldsymbol{A}^{(0)} = \frac{1}{4\pi\varepsilon_0 c^3}\frac{e^{ikR}}{R}\ddot{\boldsymbol{p}}\times\boldsymbol{e}_r, \quad \boldsymbol{E} = c\boldsymbol{B}\times\boldsymbol{e}_r = \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikR}}{R}\left(\ddot{\boldsymbol{p}}\times\boldsymbol{e}_r\right)\times\boldsymbol{e}_r. \tag{5.5}$$

Considering $\boldsymbol{p}$ along z-axis, we have

$$\boldsymbol{B} = \frac{1}{4\pi\varepsilon_0 c^3}\frac{e^{ikR}}{R}\ddot{p}\sin\theta\,\boldsymbol{e}_\phi, \quad \boldsymbol{E} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikR}}{R}\ddot{p}\sin\theta\,\boldsymbol{e}_\theta.$$

Next we discuss the energy flux density of the radiation.

$$\overline{\boldsymbol{S}} = \frac{1}{2}\Re\left(\boldsymbol{E}^*\times\boldsymbol{H}\right) = \frac{c}{2\mu_0}\Re\left(\left(\boldsymbol{B}^*\times\boldsymbol{e}_r\right)\times\boldsymbol{B}\right) = \frac{1}{32\pi^2\varepsilon_0 c^3}\frac{|\ddot{p}|^2\sin^2\theta}{R^2}\boldsymbol{e}_r. \tag{5.6}$$

$$P = \oint\left|\overline{\boldsymbol{S}}\right| R^2\mathrm{d}\Omega = \frac{1}{4\pi\varepsilon_0}\frac{|\ddot{\boldsymbol{p}}|^2}{3c^3}. \tag{5.7}$$

If $\boldsymbol{p} = \boldsymbol{p}_0 e^{-i\omega t}$, then $P = \frac{1}{4\pi\varepsilon_0}\frac{\omega^4 p_0^2}{3c^3}$, which shows that short wave radiation is much stronger.

Proposition 5.5 (Rayleigh-Jeans Formula)**.** *The power radiated by an electric dipole is*

$$P = \frac{1}{\pi\varepsilon_0}\frac{p^2}{3c^3}\omega^4 \Longrightarrow P\propto\omega^4\propto\left(\frac{1}{\lambda}\right)^4\propto f^4. \tag{5.8}$$

When the length of antenna l is much smaller than the wave length, its radiation can be approximated as an electric dipole radiation. The current is given by

$$I(z) = I_0\left(1-\frac{2}{l}|z|\right)\Longrightarrow\dot{\boldsymbol{p}} = \int_{-l/2}^{l/2}\boldsymbol{I}(z)\mathrm{d}z = \frac{1}{2}I_0\boldsymbol{l}$$

$$\Longrightarrow P = \frac{\mu_0 I_0^2\omega^2 l^2}{48\pi c} = \frac{\pi}{12}\sqrt{\frac{\mu_0}{\varepsilon_0}}I_0^2\left(\frac{l}{\lambda}\right)^2\Longrightarrow P\propto\left(\frac{l}{\lambda}\right)^2.$$

Definition 5.6. Denote $P = \frac{1}{2}R_r I_0^2$, where R_r is the **radiation resistance**.

$$R_r = 197\left(\frac{l}{\lambda}\right)^2\Omega. \tag{5.9}$$

R_r shows the ability of antenna to radiate electromagnetic wave.

5.5 Antenna Radiation

If $l \ll \lambda$, the power is approximately given by $\left(\frac{l}{\lambda}\right)^2$ or smaller. Hence, to increase the radiation power, we can increase l.

Now we consider half-wave antenna. Since

$$I(z) = I_0 \sin k\left(\frac{l}{2} - |z|\right),$$

when $l = \frac{\lambda}{2}$, the current distribution is

$$I(z) = I_0 \cos kz, \quad |z| \leqslant \frac{\lambda}{4} \Longrightarrow A_z(\boldsymbol{x}) = \frac{\mu_0 I_0}{4\pi}\int_{-\lambda/4}^{\lambda/4} \frac{e^{ikR}}{r} I_0 \cos kz \mathrm{d}z.$$

When it is a far region, we have

$$r = R - z\cos\theta \Longrightarrow A_z(\boldsymbol{x}) = \frac{\mu_0 I_0 e^{ikR}}{4\pi R}\int_{-\lambda/4}^{\lambda/4} \cos kz e^{-ikz\cos\theta}\mathrm{d}z$$

$$\Longrightarrow A_z(\boldsymbol{x}) = \frac{\mu_0 I_0 e^{ikR}}{2\pi kR}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\boldsymbol{e}_z, \quad \boldsymbol{B} = -i\frac{\mu_0 I_0}{2\pi R}e^{ikR}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\boldsymbol{e}_\phi, \tag{5.10}$$

$$\boldsymbol{E} = -i\frac{\mu_0 c I_0^2}{8\pi R}e^{ikR}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\boldsymbol{e}_\theta, \tag{5.11}$$

$$P = \frac{\mu_0 c I_0^2}{8\pi^2}\oint \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\mathrm{d}\Omega = \frac{\mu_0 c I_0^2}{4\pi^2}\int_0^\pi \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\mathrm{d}\theta = 2.44\frac{\mu_0 c I_0^2}{8\pi}. \tag{5.12}$$

Hence

$$R_r = 2.44\frac{\mu_0 c}{4\pi} = 73.2\Omega.$$

The general equation of power radiated by antenna is

$$P = \frac{\mu_0 c I_0^2}{8\pi}\left(\ln(2\pi m) - \mathrm{Ci}(2\pi m)\right), \quad \mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\mathrm{d}t, \quad l = m\frac{\lambda}{2}. \tag{5.13}$$

As the length increases, the radiation resistance increasing gets slower. Hence we often use mhalf-wave or full-wave antenna.

Example 5.7. Consider an one-dimensional antenna array N half-wave dipole antennas placed along z-axis with distance l.

Proof.

$$\boldsymbol{E}_{\text{dipole}} = \frac{1}{4\pi\varepsilon_0 c^2 R}\ddot{p}\sin\theta e^{ik\theta}\boldsymbol{e}_\theta = \frac{-i\omega I\Delta l}{4\pi\varepsilon_0 c^2 R}\sin\theta e^{ikR}\boldsymbol{e}_\theta \Longrightarrow$$

$$\begin{aligned}\boldsymbol{E} &= \sum_{m=0}^{N-1}\boldsymbol{E}_m = \sum_{m=0}^{N-1}\frac{-i\omega I\Delta l}{4\pi\varepsilon_0 c^2 R_m}\sin\theta e^{ikR_m}\boldsymbol{e}_\theta \approx \sum_{m=0}^{N-1}\frac{-i\omega I\Delta l}{4\pi\varepsilon_0 c^2 R_0}\sin\theta e^{ik(R_0 + ml\cos\theta)}\boldsymbol{e}_\theta \\ &= \boldsymbol{E}_0\sum_{m=0}^{N-1} e^{ikml\cos\theta} = \boldsymbol{E}_0\frac{1 - e^{iNkl\cos\theta}}{1 - e^{ikl\cos\theta}}.\end{aligned} \tag{5.14}$$

Hence, the radiation power is

$$|\overline{S}| = \left|\frac{1}{2}\Re\left(\boldsymbol{E}^* \times \boldsymbol{H}\right)\right| = \left|\frac{1}{2Z_0}|\boldsymbol{E}|^2\boldsymbol{n}\right| = \frac{1}{2Z_0}|\boldsymbol{E}_0|^2\left|\frac{1 - e^{iNkl\cos\theta}}{1 - e^{ikl\cos\theta}}\right|^2.$$

Denote the array factor as

$$F = \left|\frac{1 - e^{iNkl\cos\theta}}{1 - e^{ikl\cos\theta}}\right|^2 = \left|\frac{\sin\left(\frac{N}{2}kl\cos\theta\right)}{\sin\left(\frac{1}{2}kl\cos\theta\right)}\right|^2.$$

When $Nkl\cos\theta = 2m\pi$, $m \in \mathbb{Z}$, $F = 0$.

If $Nkl\cos\theta = 2\pi$, then $\theta = \arccos\frac{2\pi}{Nkl} = \arccos\frac{\lambda}{Nl}$ is called the **first null direction**. Denote $\psi = \frac{\pi}{2} - \theta$, then $\sin\psi = \frac{\lambda}{Nl}$. ψ is called the **half beam width**. Hence when $Nl \gg \lambda$, we have highly directional radiation. □

5.6 Diffraction of Electromagnetic Waves

There are two main diffractions:

- **Fresnel diffraction**: distance between the light source and the screen is finite.
- **Fraunhofer diffraction**: infinite distance.

The **Fresnel coefficient** is $F = \frac{a^2}{L\lambda}$, where a is the size of the aperture, L is the distance between the aperture and the screen.

5.7 Momentum of Electromagnetic Fields

By Maxewll equations, we have the Lorentz force density

$$\begin{aligned}\boldsymbol{f} &= \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B} = \varepsilon_0(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + \left(\frac{1}{\mu_0}\nabla\times\boldsymbol{B} - \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\times\boldsymbol{B} \\ &= \varepsilon_0(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + \frac{1}{\mu_0}(\nabla\cdot\boldsymbol{B})\boldsymbol{B} + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \varepsilon_0(\nabla\times\boldsymbol{E})\times\boldsymbol{E} - \varepsilon_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})\end{aligned}$$

Definition 5.8. The momentum density of electromagnetic field is

$$\boldsymbol{g} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B} = \varepsilon_0\mu_0\boldsymbol{E}\times\boldsymbol{H} = \frac{1}{c^2}\boldsymbol{S}. \tag{5.15}$$

$$(\nabla\cdot\boldsymbol{E})\,\boldsymbol{E} + (\nabla\times\boldsymbol{E})\times\boldsymbol{E} = \nabla\cdot(\boldsymbol{E}\boldsymbol{E}) - \frac{1}{2}\nabla\cdot\left(\overset{\leftrightarrow}{\mathscr{I}}\boldsymbol{E}^2\right)$$

Denote

$$\overset{\leftrightarrow}{\mathscr{T}} = -\varepsilon_0\boldsymbol{E}\boldsymbol{E} - \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B} + \frac{1}{2}\overset{\leftrightarrow}{\mathscr{I}}\left(\varepsilon_0\boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right)$$

$$\Longrightarrow \boldsymbol{f} + \frac{\partial\boldsymbol{g}}{\partial t} = -\nabla\cdot\overset{\leftrightarrow}{\mathscr{T}}, \quad \int_V \boldsymbol{f}\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{g}\,\mathrm{d}V = -\oint_S \overset{\leftrightarrow}{\mathscr{T}}\cdot\mathrm{d}\boldsymbol{S}. \tag{5.16}$$

Definition 5.9. $\overset{\leftrightarrow}{\mathscr{T}}$ is called the **stress tensor** or **momentum flux density tensor** of electromagnetic field.

If V is the whole space, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{g}\mathrm{d}V + \int \boldsymbol{f}\mathrm{d}V = 0.$$

For a plane wave, we have

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{e}_k \times \boldsymbol{E} \Longrightarrow \overline{\boldsymbol{g}} = \frac{1}{2}\varepsilon_0 \Re\left(\boldsymbol{E}^* \times \boldsymbol{B}\right) = \frac{\varepsilon_0}{2c}|E|^2\boldsymbol{e}_k.$$

Since $\boldsymbol{S} = cw\boldsymbol{e}_k$ for a plane wave, the momentum density is

$$\boldsymbol{g} = \frac{w}{c}\boldsymbol{e}_k.$$

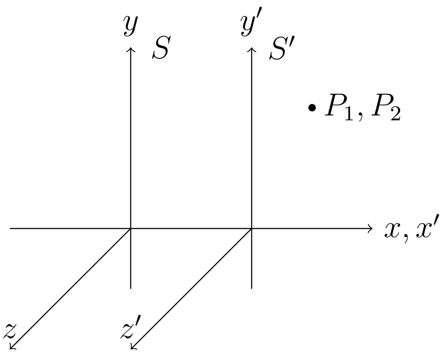
6 Special Relativity

6.1 Galileo Transformation

Definition 6.1. An **event** is the kinetic property happening in a infinitely small time interval and infinitely small space interval.

If P_1 and P_2 are two events happening at the same site in frame S and S' respectively, then

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t, \quad \dot{x}' = \dot{x} - v \iff u' = u - v, \quad \ddot{x}' = \ddot{x} \iff a' = a.$$



The time inverval between these two events is $\Delta t' = t_2' - t_1' = t_2 - t_1 = \Delta t$.

For two events happening at the same time, the space interval is $\Delta x' = x_2' - x_1' = (x_2 - vt_2) - (x_1 - vt_1) = \Delta x$. Hence, time and space are absolute in Galileo transformation.

In classical mechanics, the laws of mechanics are invariant in any inertial frame.

Definition 6.2. A law is **covariant** under a transformation if its form remains unchanged under the transformation.

Example 6.3.

$$F = m\ddot{x}, \quad \begin{cases} x' = x - vt \\ t' = t \\ m' = m \end{cases} \implies \ddot{x}' = \ddot{x} \implies F' = m'\ddot{x}'.$$

For Galileo transformation, $\boldsymbol{c}' = \boldsymbol{c} - \boldsymbol{u}$. However, by Maxwell equations, the speed of light in vacuum is constant $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ in any inertial frame, which contradicts Galileo transformation.

6.2 Experiment Fundamentals of Special Relativity

Michelson-Morley experiment shows that the speed of light is constant in any inertial frame.

$$c\Delta t \approx l\frac{v^2}{c^2} \implies \frac{2c\Delta t}{\lambda} \approx \frac{2l}{\lambda}\frac{v^2}{c^2} \approx 0.4. \tag{6.1}$$

However the experiment shows at most 0.01 fringe shift, which contradicts the above result.

$\pi^0 \to \nu + \nu$ decay experiment. π^0 moves at speed $0.9975c$, and ν moves at speed c in the rest frame of π^0. However in the lab frame, the speed of ν is c.

6.3 Basic Principles of Special Relativity and Lorentz Transformation

In 1995, Einstein proposed two basic postulates of special relativity.

Proposition 6.4 (Principle of Relativity)**.** *All inertial frames are equivalent. The laws of physics are the same in all inertial frames.*

Proposition 6.5 (Invariance of the Speed of Light)**.** *The speed of light in vacuum has the same value c in all inertial frames, independent of the motion of the light source or observer.*

There are two inertial frames Σ and Σ'. Event 1 is $(0,0,0,0)$ in both frames. Event 2 is (x,y,z,t) in Σ and (x',y',z',t') in Σ'. By the invariance of the speed of light, we have

$$x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2=0.$$

Definition 6.6. The **interval** is defined as

$$s^2=c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2. \tag{6.2}$$

If two events can be connected by a light signal, then $s^2=s'^2=0$. In general, $s^2=s'^2$.

Suppose Σ' moves along x-axis of Σ and x'-axis coincides with x-axis of Σ. We can assume

$$x'=a_{11}x+a_{12}ct,\quad y'=y,\quad z'=z,\quad ct'=a_{21}x+a_{22}ct,\quad a_{11}>0,\quad a_{22}>0.$$

$$\Longrightarrow a_{11}^2-a_{21}^2=1,\quad a_{22}^2-a_{12}^2=1,\quad a_{11}a_{12}-a_{21}a_{22}=0\Longrightarrow a_{12}=a_{21}.$$

In Σ, the origin of Σ' moves along $x=vt$ while $x'=0$. Hence,

$$0=a_{11}vt+a_{12}ct\Longrightarrow a_{11}=a_{22}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\quad a_{12}=a_{21}=\frac{-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Proposition 6.7. *The* ***Lorentz transformation*** *is*

$$\begin{cases} x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \\ y'=y \\ z'=z \\ t'=\frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \end{cases}. \tag{6.3}$$

The inverse transformation is given by substituting v with $-v$.