

Stokes Matrices and Quantum Groups

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October 18, 2026

Abstract

The subject of these notes is the Stokes phenomenon of a meromorphic linear system with a second order pole, and the observation that for the quantum confluent hypergeometric system

$$\frac{dF}{dz} = h\left(u + \frac{T}{z}\right)F$$

the Stokes matrices satisfy the *RLL* relations, hence define a representation of the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_n)$ on the same vector space on which the e_{ij} act. The first half develops the analytic machinery: formal solutions and their factorial growth, Borel–Laplace summation, canonical sectorial solutions, Stokes matrices, the connection matrix and the monodromy relation. The second half proves the *RLL* relation by means of a flat system in two variables: a compatibility computation, the formal solution in ratio coordinates, the summability and factorization theorem, and the comparison of two continuation paths. The numerical *R*-matrix is computed from the collision equation, where the relevant connection coefficient reduces to Kummer’s equation. Statements are given with full proofs where the argument is elementary or combinatorial, and with an indication of the analytic input otherwise.

Notation: $\mathfrak{h}_{\text{reg}}$ = diagonal matrices with distinct eigenvalues; $[A]$ = diagonal part of A ; $q = e^{\pi i \hbar}$; $\hbar$ = dual Cartan.

Ref	Statement	Role
2.3	unique canonical solution on each Stokes sector	analytic foundation
2.9	Stokes matrices are unipotent triangular up to permutation	makes them computable
2.13	$Ce^AC^{-1} = S_-S_+e^{2\pi i[A]}$	monodromy $\leftrightarrow$ Stokes data
4.1	Stokes matrix of the collision equation is R_0	computes R
4.10	six-region factorization of the two-variable solution	analytic input
4.15	$S_{h\pm}^{(\cdot)}$ satisfy $RLL = LLR$	proof mechanism
5.2	Stokes matrices of (3.2) give a representation of $U_q(\mathfrak{gl}_n)$ on V	main result

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1 Meromorphic linear systems and the model equation

1.1 Regular and irregular singularities

We consider a linear system with rational coefficients

$$\frac{dF}{dz} = \left(\sum_{i=1}^d \sum_{j=1}^{p_i} \frac{A_{i,j}}{(z - z_i)^j} \right) F(z), \quad F(z) \in \mathrm{GL}_n(\mathbb{C}), \quad A_{i,j} \in \mathfrak{gl}_n(\mathbb{C}), \quad (1.1)$$

with poles at $z_1, \dots, z_d$ of orders $p_1, \dots, p_d$.

Definition 1.1. A pole of order one is *regular singular*. A pole of order $p_i > 1$ is *irregular*.

Proposition 1.2. For $z_0 \in \mathbb{C} \setminus \{z_1, \dots, z_d\}$ and $G_0 \in \mathrm{GL}_n(\mathbb{C})$ there is a unique multivalued holomorphic fundamental solution of (1.1) with $F(z_0) = G_0$.

Proof. On a disc D around z_0 avoiding the poles, $A(z)$ is holomorphic and the system is solved by the convergent Peano series

$$F(z) = \left(1 + \sum_{k \geq 1} \int_{z_0}^z A(t_1) dt_1 \int_{z_0}^{t_1} A(t_2) dt_2 \cdots \right) G_0,$$

which converges uniformly on compact subsets of D because the iterated integrals are bounded by $\|A\|_\infty^k |z - z_0|^k / k!$. Analytic continuation along paths avoiding the poles produces a multivalued solution on $\mathbb{C} \setminus \{z_1, \dots, z_d\}$, and uniqueness holds because the difference of two solutions with the same initial condition solves a linear system with zero initial datum. $\square$

Continuation around the poles defines the monodromy representation $\pi_1(\mathbb{C} \setminus \{z_1, \dots, z_d\}) \rightarrow \mathrm{GL}_n(\mathbb{C})$. For a regular singular point this representation carries the local information. The purpose of the Stokes theory is that at an irregular singular point it does not, because the solutions carry an essential singularity which monodromy cannot see.

1.2 The two-point model

The model we study has one irregular point and one regular point:

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z} \right) F, \quad u \in \mathfrak{h}_{\mathrm{reg}}, \quad A \in \mathfrak{gl}_n(\mathbb{C}). \quad (1.2)$$

The results below hold for arbitrary A ; the hermitian condition assumed in some sources is not needed for the formal theory. The pole at $z = 0$ has order two and is irregular, and $z = \infty$ is a regular singular point of (1.2). This is the simplest irregular configuration, and it is the one for which all the objects below can be written down explicitly.

Proposition 1.3. The system (1.2) has a unique formal fundamental solution of the form

$$\widehat{F}(z; u, A) = (\mathrm{Id}_n + \widehat{H}(z; u, A)) e^{-u/z} z^{[A]}, \quad \widehat{H}(z) = \sum_{k \geq 1} H_k z^k, \quad (1.3)$$

where $[A]$ denotes the diagonal part of A , and whose coefficients are determined by the recursion

$$[u, H_{k+1}] = k H_k + H_k [A] - A H_k, \quad H_0 = \mathrm{Id}_n. \quad (1.4)$$

Proof. Differentiating (1.3) gives

$$\widehat{F}' = \left(\widehat{H}' + \widehat{H} \left(\frac{u}{z^2} + \frac{[A]}{z} \right) \right) e^{-u/z} z^{[A]},$$

so $\hat{F}$ solves (1.2) if and only if

$$\hat{H}' = \frac{[u, \hat{H}]}{z^2} + \frac{A\hat{H} - \hat{H}[A]}{z}. \quad (1.5)$$

Substituting $\hat{H} = \sum_{k \geq 0} H_k z^k$ and comparing the coefficient of z^{k-1} gives (1.4). We verify that (1.4) determines the H_k uniquely.

At $k = 0$, (1.4) reads $[u, H_1] = H_0[A] - AH_0 = [A] - A$, whose diagonal part is 0. Hence $A - [A]$ has zero diagonal, as it must, and no condition is imposed on $[A]$. The off-diagonal part of $[u, H_1]$ is $(u_i - u_j)(H_1)_{ij}$, so

$$(H_1)_{ij} = -\frac{A_{ij}}{u_i - u_j}, \quad i \neq j, \quad (1.6)$$

and $(H_1)_{ii}$ is still free.

Suppose $H_1, \dots, H_k$ are known. Taking the (i, j) entry of (1.4) with $i \neq j$ gives

$$(u_i - u_j)(H_{k+1})_{ij} = k(H_k)_{ij} + (H_k[A])_{ij} - (AH_k)_{ij},$$

which determines $(H_{k+1})_{ij}$ by division by the nonzero gap $u_i - u_j$. The diagonal part ($i = j$) gives

$$0 = k(H_k)_{ii} + (H_k[A])_{ii} - (AH_k)_{ii},$$

an identity which holds automatically for the already determined H_k ; it imposes no new condition but will be used below. Finally, the diagonal entries $(H_{k+1})_{ii}$ are not touched by $[u, H_{k+1}]$, so they are free at this stage and are fixed by the diagonal part of (1.4) at the next step: replacing k by $k + 1$ and taking $i = j$,

$$(k + 1)(H_{k+1})_{ii} = (AH_{k+1} - H_{k+1}[A])_{ii} = \sum_{j \neq i} A_{ij}(H_{k+1})_{ji}, \quad (1.7)$$

since $[A]$ is diagonal. The right hand side involves only the off-diagonal part of H_{k+1} , which was determined in the previous paragraph. Hence $(H_{k+1})_{ii}$ is determined by (1.7), and the induction closes. The case $k = 0$ is the statement that $(H_1)_{ii}$ is determined by (1.7) with $k = 0$, i.e. $(H_1)_{ii} = \sum_{j \neq i} A_{ij}(H_1)_{ji}$. $\square$

Remark 1.4. The exponent $[A]$ in (1.3) is forced rather than chosen: the diagonal part of (1.4) at $k = 0$ reads $0 = H_0[A] - AH_0$, i.e. $[A]$ is the diagonal part of A . Replacing A by a conjugate changes $[A]$ to the diagonal part of the conjugate, and the two normalizations differ by a constant gauge. This is the sense in which the formal type is an invariant.

Example 1.5. For $n = 1$, $u = u_1$ and $A = A_{11}$ are scalars, and (1.2) reads $F' = (u/z^2 + A/z)F$ with solution $F = z^A e^{-u/z}$. The formal solution (1.3) is this exact solution, and $\hat{H} \equiv 1$. So the divergence of $\hat{H}$ in higher rank is a genuinely noncommutative phenomenon.

Example 1.6. For $n = 2$, $u = \text{diag}(u_1, u_2)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, (1.6) gives

$$H_1 = \begin{pmatrix} h_{11} & -\frac{b}{u_1 - u_2} \\ -\frac{c}{u_2 - u_1} & h_{22} \end{pmatrix}, \quad h_{11} = \frac{bc}{(u_1 - u_2)^2}, \quad h_{22} = \frac{bc}{(u_1 - u_2)^2},$$

where the diagonal entries come from (1.7) at $k = 0$. Already at order one the coefficients depend rationally on the spectral gaps and quadratically on the entries of A .

1.3 Factorial growth of the coefficients

The formal solution (1.3) usually diverges. What the recursion (1.4) does provide is a precise rate of growth.

Lemma 1.7 (Gevrey bound). *There are constants $c > 0$ and $K > 0$ with*

$$\|H_m\| \leq cK^m m!, \quad m \geq 0. \quad (1.8)$$

Proof. Fix any norm on $\mathfrak{gl}_n(\mathbb{C})$ with $\|XY\| \leq \|X\|\|Y\|$, and put

$$g = \min_{i \neq j} |u_i - u_j| > 0, \quad \alpha = \max\{\|A\|, g^{-1}\}.$$

By (1.4) and (1.7), there is a constant β depending only on n and g with

$$\|H_{k+1}\| \leq \beta(k+1)\alpha\|H_k\|.$$

Indeed the off-diagonal entries of H_{k+1} are bounded by $g^{-1}(k\|H_k\| + 2\|A\|\|H_k\|)$, and the diagonal entries by the same expression through (1.7). Iterating,

$$\|H_m\| \leq \beta^m \alpha^m m! \|H_0\| = \beta^m \alpha^m m!,$$

which is (1.8) with $K = \alpha\beta$. □

The estimate (1.8) is sharp in general. The coefficients of a formal solution of an irregular system at a pole of order two grow like a constant to the power m times $m!$, and this is exactly the class of series that Borel summation can handle.

1.4 Borel–Laplace summation

Definition 1.8. Let $\hat{f}(z) = 1 + \sum_{p \geq 1} a_p z^{-p}$ be a formal power series in z^{-1} . Its *Borel transform* is the formal power series

$$\mathcal{B}(\hat{f})(\xi) = \sum_{p \geq 0} \frac{a_p}{p!} \xi^p.$$

For g holomorphic near $\xi = 0$ and a direction d , the *Laplace transform of g along d* is

$$(\mathcal{L}_d g)(z) = 1 + \int_0^{\infty \cdot e^{id}} e^{-z\xi} g(\xi) d\xi,$$

the integration being along the ray $\arg \xi = d$.

By lemma 1.7, the Borel transform of $\hat{H}$ is the convergent germ

$$g(\xi) := \mathcal{B}(\hat{H})(\xi) = \sum_{k \geq 1} \frac{H_k}{(k-1)!} \xi^{k-1}, \quad (1.9)$$

where we used $k!/k! = 1$ and $1/k! = (1/k) \cdot 1/(k-1)!$; more precisely $\mathcal{B}(H_k z^k) = H_k \xi^k / k! = H_k \xi^{k-1} / (k-1)! \cdot (1/k) \cdot k$, and the normalization $(k-1)!$ is the one that makes the Laplace integral below reproduce the series without an extra factor. The radius of convergence of (1.9) is at least $1/K$.

Lemma 1.9. *For $h \notin \mathbb{Q}$ and an admissible direction d (see definition 2.1) there is $\varepsilon > 0$ such that g continues analytically to the sector $-d + \varepsilon < \arg \xi < -d - \varepsilon$, and there are $\alpha, \beta > 0$ with $|g(\xi)| \leq \alpha e^{\beta|\xi|}$ there.*

Proof. Multiplying (1.4) by z^{k+1} and summing against the kernel of the Borel transform, the recursion is formally equivalent to the integral equation

$$g(\xi)u - (u + \xi \text{Id}_n)g(\xi) = A - [A] + \int_0^\xi (Ag(t) - g(t)[A])dt, \quad (1.10)$$

which we now read as a differential equation for g along a ray. Its off-diagonal part is a linear first order system with a regular singular point at $\xi = u_j - u_i$ and no other singularity, so g continues along any ray which does not pass through one of the points $u_j - u_i$. The rays through the points $u_j - u_i$ have directions $\arg(u_j - u_i)$, and the sector of continuation of g is the reflection of the sector of validity of $\mathcal{L}_d$, which is why the condition on d in [definition 2.1](#) involves $-\arg(u_i - u_j)$. The diagonal part of (1.10) determines the scalar factor of the solution and produces the bounds on the growth; iterating (1.10) with the growth ansatz $g = \alpha e^{\beta|\xi|}$ closes for β large enough, because the integral adds one power of ξ and the factorial in the kernel has been absorbed by the Borel normalization. $\square$

Lemma 1.10. *There are $C, D > 0$ such that for all $N \geq 1$ and all z with $|z| > \beta$ and $-d - \frac{\pi}{2} < \arg z < -d + \frac{\pi}{2}$,*

$$\left| \mathcal{L}_d(\mathcal{B}\hat{H})(z) - \sum_{m=0}^{N-1} H_m z^{-m} \right| \leq C N^N e^{-N} \frac{|z|^N}{D^N}. \quad (1.11)$$

Proof. Insert the Taylor expansion of g up to order $N - 1$ with integral remainder into the Laplace integral and use the term-by-term identity

$$\int_0^{\infty e^{id}} e^{-z\xi} \xi^{m-1} d\xi = (m-1)! z^m, \quad (1.12)$$

valid for $\text{Re}(e^{id}/z) > 0$, i.e. for $|\arg z - d| < \pi/2$. This recovers $\sum_{m \leq N-1} H_m z^{-m}$ exactly, and the remainder is the Laplace transform of the Taylor remainder, which is bounded by the exponential growth bound of [lemma 1.9](#). The factor $N^N e^{-N}$ is Stirling's estimate for $N!(m/N)^N$ after optimizing over N . $\square$

Corollary 1.11. *The function $F_d(z) := \mathcal{L}_d(\mathcal{B}\hat{H})(z) \cdot z^{[A]} e^{-u/z}$ is a solution of (1.2), holomorphic for $|z| > \beta$ in the sector $|\arg z - d| < \pi/2$, and $\hat{F}$ is its asymptotic expansion there.*

Proof. The Borel and Laplace transforms commute with $\frac{d}{dz}$ and with multiplication by z^{-1} , so $\mathcal{L}_d(\mathcal{B}\hat{H})$ satisfies (1.5) and F_d satisfies (1.2). The asymptotic statement is [lemma 1.10](#). $\square$

Remark 1.12. The content of [lemma 1.9–corollary 1.11](#) is that the divergent series $\hat{H}$ is nevertheless the asymptotic expansion of an honest function, and that the function is produced by an integral whose contour is a choice. The freedom in that choice is exactly the Stokes phenomenon: two different admissible directions may give different functions with the same asymptotics.

2 Stokes phenomenon and monodromy data

2.1 Anti-Stokes rays

Definition 2.1. The *anti-Stokes rays* of (1.2) are the directions along which one of the exponential factors $e^{-(u_i - u_j)/z}$ decays fastest as $z \rightarrow 0$. Their set is

$$\text{aSR}(u) = \{\arg(u_i - u_j) + 2k\pi : k \in \mathbb{Z}, i \neq j\}.$$

A direction d is *admissible* if $d \notin \text{aSR}(u)$.

The name is justified as follows. Writing $u_i - u_j = \rho e^{i\theta}$ and $z = r e^{i\phi}$,

$$|e^{-(u_i - u_j)/z}| = \exp\left(-\frac{\rho}{r} \cos(\theta - \phi)\right),$$

so that the factor decays as $r \rightarrow 0$ exactly when $\cos(\theta - \phi) > 0$, i.e. $|\phi - \theta| < \frac{\pi}{2}$, and decays fastest when $\phi = \theta$. Combined with $e^{-u_i/z}$ versus $e^{-u_j/z}$, the distinguished directions are the $\theta = \arg(u_i - u_j)$, up to 2π .

Definition 2.2. Fix an initial anti-Stokes direction τ_0 and arrange the others in increasing order $\dots < \tau_{-1} < \tau_0 < \tau_1 < \dots$. For an admissible direction $d \in (\tau_j, \tau_{j+1})$ the *Stokes sector* is

$$\text{Sect}_d = \left\{ z \in \overline{\mathbb{C}} : \arg z \in \left(\tau_j - \frac{\pi}{2}, \tau_{j+1} + \frac{\pi}{2}\right) \right\}.$$

Each Stokes sector has opening angle strictly between π and 2π , and Sect_d is the largest sector on which the ordering of the exponential factors $e^{-u_i/z}$ is constant, with $e^{-u_i/z}$ dominating $e^{-u_j/z}$ precisely when $\text{Re}((u_j - u_i)/z) > 0$.

Theorem 2.3 (Canonical solutions). *For each $i \in \mathbb{Z}$ there is a unique holomorphic fundamental solution $F_i(z)$ of (1.2) with the prescribed asymptotics*

$$F_i(z) \cdot e^{-u/z} z^{-[A]} \rightarrow \text{Id}_n \quad \text{as } z \rightarrow 0 \text{ within } \text{Sect}(\tau_i - \frac{\pi}{2}, \tau_{i+1} + \frac{\pi}{2}). \quad (2.1)$$

Proof. Existence follows from [corollary 1.11](#) applied to an admissible direction $d \in (\tau_i, \tau_{i+1})$, which produces a solution with the asymptotics (2.1) on the subsector $|\arg z - d| < \pi/2$. By [lemma 3.6](#) below, or rather its classical analogue, the sum does not depend on the choice of d inside (τ_i, τ_{i+1}) ; hence it extends to the whole enlarged sector $\text{Sect}(\tau_i - \frac{\pi}{2}, \tau_{i+1} + \frac{\pi}{2})$ by analytic continuation. This is the content of the enlargement step: the individual Borel sums along a ray only converge on a half-plane, and the claim is that the resulting functions agree on the overlaps, so that they glue.

For uniqueness, suppose F and $\tilde{F}$ both satisfy (2.1) on the sector, and let $G = \tilde{F}F^{-1}$. Then G is holomorphic in the sector and $G \rightarrow \text{Id}_n$ as $z \rightarrow 0$ there. On the other hand G is a ratio of two fundamental solutions, so $\frac{dG}{dz} = 0$ and G is constant. Comparing limits along a ray in the sector gives $G = \text{Id}_n$. $\square$

Remark 2.4. The uniqueness statement of [theorem 2.3](#) is thus a statement about the pair (sector, prescribed normalization), not about a limit along a single ray: along a single ray many functions share the same asymptotics, since $e^{-a/z}$ is flat there for $a > 0$. The tool that upgrades an asymptotic equivalence on a sector to an identity is the following.

Lemma 2.5 (Watson). *Let F be holomorphic on a sector Sect and suppose $F(z)e^{-u/z}z^{-[A]} \rightarrow \text{Id}_n$ as $z \rightarrow 0$ within Sect . If F is a fundamental solution of (1.2), it is unique with this property.*

Proof. If $F, \tilde{F}$ are two such, then $G = \tilde{F}F^{-1}$ has $G' = 0$ hence is constant, and $G \rightarrow \text{Id}_n$ by the prescribed asymptotics. $\square$

Remark 2.6. Enlarging to a full Stokes sector of opening $> \pi$ is what gives [lemma 2.5](#) enough room to apply, and it is also what makes the sectors overlap so that transition matrices between different canonical solutions can be defined. On a half-plane sector, two different Borel sums can agree asymptotically without being equal.

2.2 Stokes matrices

Definition 2.7. For an anti-Stokes direction τ_i the associated *Stokes factor* is defined by

$$F_{i+1}(z) = F_i(z) \cdot S_{\tau_i}(u). \quad (2.2)$$

The matrix S_{τ_i} is constant: both sides solve (1.2), so $F_i^{-1}F_{i+1}$ has zero logarithmic derivative, exactly as in the proof of theorem 2.3.

Definition 2.8. For $u \in \mathfrak{h}_{\text{reg}}$ and an admissible direction $d \in (\tau_i, \tau_{i+1})$ the *Stokes matrix* $S_d(u) \in \text{GL}_n(\mathbb{C})$ of (1.2) is determined by

$$F_{d+\pi}(z) = F_d(z) \cdot S_d(u). \quad (2.3)$$

Since $d + \pi$ lies in the next Stokes sector, S_d is the ordered product of the Stokes factors attached to the anti-Stokes rays between d and $d + \pi$. Writing $F_d = F_i$, $F_{d+\pi} = F_j$ and inserting the intermediate canonical solutions,

$$S_d(u) = S_{\tau_i}(u) S_{\tau_{i+1}}(u) \cdots S_{\tau_{j-1}}(u). \quad (2.4)$$

This factorization is the reason one can work with the small factors S_{τ_i} , each of which has a single nonzero off-diagonal block, instead of S_d .

Lemma 2.9 (Triangularity). *Let J be the permutation matrix of the dominance ordering of $\{e^{u_1/z}, \dots, e^{u_n/z}\}$ along the direction $d + \frac{\pi}{2}$; that is, $J_{ij} = \delta_{\sigma(i)j}$ where $\sigma(i) < \sigma(j)$ if and only if $e^{(u_i - u_j)/z} \rightarrow 0$ as $z \rightarrow 0$ along $d + \frac{\pi}{2}$. Then $J^{-1}S_d(u)J$ is upper triangular, $J^{-1}S_{d+\pi}(u)J$ is lower triangular, and*

$$S_d(u)_{kk}^{-1} = S_{d+\pi}(u)_{kk} = 1 \in \mathbb{C}, \quad k = 1, \dots, n.$$

Proof. By (2.3),

$$e^{-u/z} z^{[A]} S_d(u) e^{u/z} z^{-[A]} = F_d(z) \cdot e^{-u/z} z^{-[A]} \left(F_{d+\pi}(z) e^{-u/z} z^{-[A]} \right)^{-1}$$

where the right hand side is a product of two matrices which both tend to Id_n within $\text{Sect}_d \cap \text{Sect}_{d+\pi}$, because in that intersection both asymptotics (2.1) hold. Hence the left hand side also tends to Id_n as $z \rightarrow 0$ in $\text{Sect}_d \cap \text{Sect}_{d+\pi}$.

The left hand side has entries

$$e^{-(u_i - u_j)/z} (z^{[A]} S_d(u) z^{-[A]})_{ij} = e^{-(u_i - u_j)/z} e^{(a_i - a_j) \log z} S_{d,ij}, \quad [A] = \text{diag}(a_1, \dots, a_n),$$

where we used that S_d is a constant matrix. A logarithmic factor cannot affect the conclusion, so if $e^{-(u_i - u_j)/z}$ does not tend to 0 along $d + \frac{\pi}{2}$, then the only way the entry can be bounded is $S_{d,ij} = 0$. Since $d + \frac{\pi}{2} \in \text{Sect}_d \cap \text{Sect}_{d+\pi}$, this determines which entries vanish, and after conjugation by J they are exactly the entries below the diagonal. For $i = j$ the entry tends to $S_{d,ii}$, so $S_{d,ii} = 1$. The argument for $S_{d+\pi}$ is the same, with $d + \frac{3\pi}{2}$ in place of $d + \frac{\pi}{2}$, and gives the opposite triangularity. $\square$

Lemma 2.10. $S_{d+2\pi}(u) = e^{2\pi i[A]} S_d(u) e^{-2\pi i[A]}$ for every admissible d .

Proof. Continuing F_d once around the origin replaces the branch of $\log z$ by $\log z + 2\pi i$, hence

$$F_{d+2\pi}(z) = F_d(z) e^{2\pi i[A]}.$$

On the other hand, applying (2.3) twice,

$$F_{d+2\pi} = F_{d+\pi} S_{d+\pi} = F_d S_d S_{d+\pi}.$$

Comparing,

$$S_d(u)S_{d+\pi}(u) = e^{2\pi i[A]}. \quad (2.5)$$

Now replace d by $d + \pi$ in (2.3): since the equation for $F_{d+2\pi}$ in terms of $F_{d+\pi}$ is the same as that for F_d in terms of $F_{d-\pi}$, and $F_{d-\pi} = F_d S_{d-\pi}$, one gets $S_{d+2\pi} = e^{2\pi i[A]} S_d e^{-2\pi i[A]}$ by conjugating (2.5). $\square$

By lemma 2.10, the pair $(S_d, S_{d+\pi})$ determines all the others, so it suffices to study one pair. From now on we choose the initial direction so that the permutation matrix is the identity, and write

$$S_+(u) := S_d(u) \text{ upper triangular}, \quad S_-(u) := S_{d+\pi}(u) \text{ lower triangular}. \quad (2.6)$$

By lemma 2.10 and (2.5) the two are inverse up to a diagonal conjugation:

$$S_+(u)S_-(u) = e^{2\pi i[A]}. \quad (2.7)$$

2.3 Connection matrix and the monodromy relation

At the regular singular point $z = 0$, the local theory gives a second canonical solution.

Lemma 2.11. *Assume (1.2) is non-resonant, i.e. no two eigenvalues of A differ by a positive integer. Then there is a unique multivalued holomorphic fundamental solution $F_0(z; u, A)$ with*

$$F_0(z; u, A) \cdot z^{-A/2\pi i} \rightarrow \text{Id}_n \quad (z \rightarrow 0).$$

Proof. Write $F_0 = G(z)z^{A/2\pi i}$ with G holomorphic and $G(0) = \text{Id}_n$. Substitution into (1.2) gives the recursion

$$(m \text{Id} + \frac{1}{2\pi i} \text{ad}_A)G_m = -uG_{m-1},$$

whose operator is invertible precisely when A has no two eigenvalues differing by a positive integer, which is the non-resonance hypothesis. The inverse is $O(m^{-1})$ for large m , so the series for G converges and the solution is unique. $\square$

Definition 2.12. The *connection matrix* $C(u, A) \in \text{GL}_n(\mathbb{C})$, taken with respect to the chosen initial anti-Stokes direction, is determined by

$$F_0(z; u, A) = F_+(z; u, A) \cdot C(u, A). \quad (2.8)$$

The matrix C is constant because both sides solve (1.2).

Proposition 2.13 (Monodromy relation). *For $u \in \mathfrak{h}_{\text{reg}}$,*

$$C(u, A)e^A C(u, A)^{-1} = S_-(u, A)S_+(u, A)e^{2\pi i[A]}. \quad (2.9)$$

Proof. A simple negative loop around 0 is a simple positive loop around ∞ , and the contribution of the two ends must agree. Concretely, continue F_0 along a negative loop: the branch of $z^{A/2\pi i}$ acquires the factor e^{-A} , so

$$F_0(ze^{-2\pi i}) = F_0(z)e^{-A}.$$

On the other hand $F_0 = F_+C$, and continuing F_+ negatively to F_- passes through the intermediate sector, so the same loop, read in the canonical solutions, is

$$F_0(ze^{-2\pi i}) = F_-(z)S_+e^{2\pi i[A]}C = F_0C^{-1}S_+e^{2\pi i[A]}C.$$

Comparing the two expressions for $F_0(ze^{-2\pi i})$ and using that F_0 is invertible gives $e^{-A} = C^{-1}S_+S_-e^{2\pi i[A]}C$, which after rearrangement is (2.9). $\square$

Remark 2.14. Equation (2.9) says that the monodromy of the connection, which is the product $S_- S_+ e^{2\pi i[A]}$ read in the frame C , is the regular-singular monodromy e^A . Equivalently, in the frame of the canonical solutions the monodromy is $e^{-2\pi i[A]}$ times a product of Stokes matrices: the irregularity of the singularity has been converted into the noncommutativity of the Stokes factors, and C measures the discrepancy. This is the irregular Riemann–Hilbert correspondence in its simplest form: the pair (S_+, S_-) together with the formal data determines the analytic connection, and the connection matrix is the change of frame realizing it.

2.4 Worked example: the Euler equation

For $n = 2$ the whole construction can be computed by hand, which is useful before turning to the operator-valued case. Take

$$\frac{dF}{dz} = \left[\frac{1}{z^2} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right] F. \quad (2.10)$$

A column of the form $(zf, z^2 f')^\top$ reduces (2.10) to the scalar equation

$$z^2 f' + (1 + z)f = 1. \quad (2.11)$$

Indeed, if $F^{(1)} = (zf, z^2 f')^\top$, then $F^{(1)'} = (f + zf', 2zf' + z^2 f'')^\top$ while the right hand side of (2.10) is $\left(\frac{f+zf'}{z} + \frac{zf+z^2 f'}{z}, -\frac{z^2 f'}{z^2} - \frac{zf+z^2 f'}{z} \right)^\top = (f + zf' + (f + zf'), -f' - f - zf')^\top$, and the two agree exactly when (2.11) holds.

For a formal solution $f = \sum_{m \geq 0} a_m z^m$, comparing coefficients gives $a_0 = 1$ and $a_m = -ma_{m-1}$, hence

$$f = \sum_{m \geq 0} (-1)^m m! z^m, \quad (2.12)$$

a series of zero radius of convergence whose coefficients have exactly the Gevrey growth (1.8). Its Borel sum along an admissible direction is

$$f_d(z) = \frac{1}{z} \int_0^{\infty e^{id}} \frac{e^{-\xi/z}}{1 + \xi} d\xi, \quad (2.13)$$

because expanding $(1 + \xi)^{-1} = \sum_{m \geq 0} (-\xi)^m$ and using (1.12) term by term returns (2.12). Integration by parts verifies (2.11): with $w = 1 + \xi$,

$$z^2 f'_d + f_d = \frac{1}{z} \int_0^{\infty e^{id}} \frac{\xi e^{-\xi/z}}{1 + \xi} d\xi = \frac{1}{z} \int_0^{\infty e^{id}} e^{-\xi/z} d\xi - \frac{1}{z} \int_0^{\infty e^{id}} \frac{e^{-\xi/z}}{1 + \xi} d\xi = 1 - f_d,$$

which is (2.11). The only finite singularity of the integrand is the pole at $\xi = -1$, so the admissible contours are those avoiding the negative real ray, and the anti-Stokes rays are $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$.

A second solution of (2.10) is exact, namely $(-e^{1/z}, (1 + z^{-1})e^{1/z})^\top$, so

$$F_d(z) = \begin{pmatrix} z f_d(z) & -e^{1/z} \\ z^2 f'_d(z) & (1 + z^{-1})e^{1/z} \end{pmatrix}, \quad \det F_d = e^{1/z}, \quad (2.14)$$

the determinant statement being exactly (2.11) rearranged as $(1 + z)f_d + z^2 f'_d = 1$.

Now compare the two lateral sums. Taking contours whose difference winds once positively around $\xi = -1$, the residue theorem gives

$$f_{d=-\pi+\varepsilon}(z) - f_{d=-\pi-\varepsilon}(z) = \frac{2\pi i}{z} e^{1/z},$$

since $\text{Res}_{\xi=-1}(e^{-\xi/z}/(1+\xi)) = e^{1/z}$. The first column of (2.14) therefore changes by $-2\pi i$ times the second column, while the second column is single valued. Hence

$$F_{d=-\pi+\varepsilon} = F_{d=-\pi-\varepsilon} \begin{pmatrix} 1 & 0 \\ -2\pi i & 1 \end{pmatrix}. \quad (2.15)$$

The transition matrix is unipotent lower triangular, in agreement with lemma 2.9, and its off-diagonal entry is computed by a residue. The formal series is the same on both sides; only its two lateral analytic realizations differ, and they differ by an exponentially small solution $e^{1/z}$ relative to the dominant behaviour. This is the Stokes phenomenon in the case where nothing has to be estimated.

2.5 The 2×2 Kummer computation

The general rank two case is the local model of the collision equation of section 4, so we record the answer. Consider

$$\frac{dF}{dz} = \left(\frac{1}{z^2} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) F, \quad u_2 - u_1 = \rho e^{i\theta}, \quad -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}. \quad (2.16)$$

One fundamental solution is given by confluent hypergeometric functions, $Y(z) = [Y_{ij}]$ with

$$Y_{11} = \frac{1}{2} e^{\lambda_1 z} \left(\frac{u_2 - u_1}{z} \right)^{a_{11} + \beta} {}_1F_1 \left(\beta, \beta - \alpha + 1; \frac{u_1 - u_2}{z} \right), \quad (2.17)$$

and the other three entries obtained from it by the Kummer transformations, where $\alpha = \lambda_1 - a_{11}$, $\beta = \lambda_2 - a_{11}$ and λ_1, λ_2 are the eigenvalues of the residue matrix. The anti-Stokes rays are $\theta, \theta \pm \pi$, and the sectors are $\text{Sect}(\theta - \frac{\pi}{2}, \theta + \frac{3\pi}{2})$ and $\text{Sect}(\theta - \frac{3\pi}{2}, \theta + \frac{\pi}{2})$. Using the standard asymptotics of ${}_1F_1$ recalled in lemma 2.15 below, one finds $Y \sim \hat{F} D U_1$ on the first sector and $Y \sim \hat{F} D U_2$ on the second, with D diagonal, and therefore

$$S_- = F_2^{-1} F_1 = D U_2 U_1^{-1} D^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{2\pi i a_{12} (u_2 - u_1)^{a_{22} - a_{11}} e^{i\pi(a_{11} - a_{22})}}{\Gamma(1 + \alpha) \Gamma(1 + \beta)} & 1 \end{pmatrix}. \quad (2.18)$$

The essential point for us is the denominator: the Stokes multiplier is a product of reciprocal Gamma functions of the residue exponents. This is the mechanism by which, in the collision equation below, the answer turns out to depend on h only through $q - q^{-1}$.

Lemma 2.15. *For $|\arg z| \leq \frac{\pi}{2}$,*

$${}_1F_1(a; b; z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{1/z} z^{b-a} [1 + O(z)] + \frac{\Gamma(b)}{\Gamma(b-a)} e^{\pm i\pi a} z^a [1 + O(z)],$$

the upper sign for $-\frac{3\pi}{2} < \arg z < \frac{\pi}{2}$ and the lower sign for $\frac{\pi}{2} \leq \arg z < \frac{3\pi}{2}$. The primary Stokes lines are $\arg z = 0$ and $\arg z = \pi$.

3 The quantum confluent hypergeometric equation

3.1 Setting

Let $T = (T_{ij})$ be the $n \times n$ matrix with entries in the universal enveloping algebra $U(\mathfrak{gl}_n)$,

$$T_{ij} = e_{ij}, \quad 1 \leq i, j \leq n, \quad [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}. \quad (3.1)$$

For a finite-dimensional irreducible $\mathfrak{gl}_n$ -module V with highest weight λ , consider

$$\frac{dF}{dz} = h\left(u + \frac{T}{z}\right)F, \quad F(z) \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n), \quad (3.2)$$

where $h \in \mathbb{C}$ is a parameter and $u \in \mathfrak{h}_{\text{reg}}$ acts as a scalar matrix on V and by scalar entries in $U(\mathfrak{gl}_n)$. The action of the coefficient matrix on $F(z)$ is matrix multiplication in the two $\text{End}(\mathbb{C}^n)$ factors together with the representation of $\mathfrak{gl}_n$ on V . This system is the quantum confluent hypergeometric equation of [6]; it is the operator-valued analogue of the model (1.2), with the irregular point at $z = \infty$ instead of $z = 0$, which is a change of variable and of sign convention only.

Example 3.1. For $\mathfrak{gl}_2$ and $V = \mathbb{C}^2$ the dual natural representation, T is the 2×2 matrix with entries e_{ij} acting on the four-dimensional space $\text{End}(V) \otimes \text{End}(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C}^2$, and e_{11} and e_{22} act by -1 and 0 in a suitable basis. The system becomes

$$\frac{dF}{dz} = \frac{1}{z^2} \begin{pmatrix} hu_1 & 0 & 0 & 0 \\ 0 & hu_1 & 0 & 0 \\ 0 & 0 & hu_2 & 0 \\ 0 & 0 & 0 & hu_2 \end{pmatrix} F + \frac{1}{z} \begin{pmatrix} 0 & -h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -h & 0 \end{pmatrix} F, \quad (3.3)$$

which decouples into two 1×1 equations with trivial Stokes matrices and one 2×2 system

$$\frac{dF}{dz} = \frac{1}{z^2} \begin{pmatrix} hu_1 & 0 \\ 0 & hu_2 \end{pmatrix} F + \frac{1}{z} \begin{pmatrix} 0 & -h \\ -h & 0 \end{pmatrix} F.$$

This is exactly the rank two system (2.16) with residue matrix $\begin{pmatrix} 0 & -h \\ -h & 0 \end{pmatrix}$, whose eigenvalues are $\pm h$. It is the reason the numerical R -matrix below is computed from Kummer's equation.

3.2 Non-resonant case

Proposition 3.2. For $h \notin \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, (3.2) has a unique formal fundamental solution

$$\hat{F}(z) = \hat{H}(z)e^{hu_1 z} z^{h\delta T}, \quad \hat{H}(z) = 1 + H_1 z^{-1} + H_2 z^{-2} + \cdots, \quad \delta T = \sum_k e_{kk} \otimes E_{kk}, \quad (3.4)$$

with all $H_m \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n)$, determined by

$$[H_{m+1}, u] = \left(\frac{m}{h} + T\right)H_m - H_m \delta T. \quad (3.5)$$

Proof. Substituting (3.4) into (3.2) and using that δT commutes with u gives, for $\hat{H}$,

$$\frac{1}{h} \frac{d\hat{H}}{dz} + \hat{H} \left(u + \frac{\delta T}{z}\right) = \left(u + \frac{T}{z}\right) \hat{H}, \quad (3.6)$$

and comparing the coefficients of z^{-m-1} yields (3.5). Write $H_m = \sum_{k,l} H_{m,kl} \otimes E_{kl}$ with $H_{m,kl} \in \text{End}(V)$. Comparing the coefficient of E_{kl} in (3.5), and using $T = \sum e_{kl} \otimes E_{kl}$ and $\delta T = \sum e_{kk} \otimes E_{kk}$, gives for $k \neq l$

$$(u_l - u_k)H_{m+1,kl} = \frac{m}{h} H_{m,kl} + \sum_j e_{kj} H_{m,jl} - H_{m,kl} e_{ll}, \quad (3.7)$$

and, replacing m by $m+1$ and taking $k = l$,

$$0 = \sum_{j \neq k} e_{kj} H_{m+1,jk} + \frac{m+1}{h} H_{m+1,kk} + [e_{kk}, H_{m+1,kk}]. \quad (3.8)$$

Since $u_k \neq u_l$, (3.7) determines the off-diagonal entries of H_{m+1} from H_m by division by the nonzero scalar gap $u_l - u_k$. The remaining equation (3.8) must determine the diagonal entries $H_{m+1,kk}$; it is an equation of the form

$$\left(\frac{m+1}{h}\text{Id} + \text{ad}_{e_{kk}}\right)X = -\sum_{j \neq k} e_{kj}H_{m+1,jk}$$

in the unknown $X = H_{m+1,kk} \in \text{End}(V)$, and it has a unique solution provided the operator $\frac{m+1}{h}\text{Id} + \text{ad}_{e_{kk}}$ is invertible on $\text{End}(V)$.

It remains to check invertibility for $h \notin \mathbb{Q}$. On a finite-dimensional irreducible $\mathfrak{gl}_n$ -module the eigenvalues of e_{kk} are the weights, and any two weights of V differ by an integer combination of the simple roots; the eigenvalues of $\text{ad}_{e_{kk}}$ on $\text{End}(V)$ are therefore integers r . The operator $\frac{m+1}{h}\text{Id} + \text{ad}_{e_{kk}}$ fails to be invertible exactly when $\frac{m+1}{h} + r = 0$ for some integer r , i.e. $h = -\frac{m+1}{r} \in \mathbb{Q}$. So for $h \notin \mathbb{Q}$ it is invertible for every $m \geq 0$, and the induction determines all H_m uniquely. $\square$

Remark 3.3. The hypothesis $h \notin \mathbb{Q}$ enters only through the invertibility above. It is sufficient but not necessary: whether the operator can be singular depends on which integers actually occur as weights of V . For V one dimensional all the $\text{ad}_{e_{kk}}$ vanish and there is no resonance at all. When resonance does occur the formal solution ceases to be unique, and the correct statement is that all formal solutions are obtained from one distinguished solution by a finite-dimensional family of gauges; we return to this in [section 5.3](#).

Definition 3.4. The anti-Stokes rays of (3.2) are

$$\text{aSR}(hu) = \{-\arg(u_i - u_j) + 2k\pi : k \in \mathbb{Z}, i \neq j\},$$

and d is admissible if $d \notin \text{aSR}(hu)$. The Stokes sector Sect_d is defined as in [definition 2.2](#).

The sign in [definition 3.4](#) is opposite to [definition 2.1](#) because the irregular point has been moved from $z = 0$ to $z = \infty$: the relevant exponential is $e^{h(u_i - u_j)z}$, which decays as $z \rightarrow \infty$ along a direction ϕ when $\text{Re}(h(u_i - u_j)e^{i\phi}) < 0$.

Proposition 3.5 (Canonical solutions). *For $h \notin \mathbb{Q}$, $u \in \mathfrak{h}_{\text{reg}}$ and an admissible direction d there is a unique $\text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ -valued holomorphic fundamental solution $F_d(z)$ on the universal covering with*

$$F_d(z)z^{-\delta T}e^{-uz} \sim 1, \quad z \rightarrow \infty \text{ within } \text{Sect}_d. \quad (3.9)$$

Proof. Same as [theorem 2.3](#): apply Borel–Laplace summation along d to $\widehat{H}$, using [lemma 1.9](#), [lemma 1.10](#) and [corollary 1.11](#) verbatim with $\text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ in place of $\mathfrak{gl}_n(\mathbb{C})$. The operator norm estimates are uniform because $\text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ is finite dimensional. $\square$

Lemma 3.6. *For admissible $\tau_i < d_1, d_2 < \tau_{i+1}$ one has $\mathcal{L}_{d_1}(\mathcal{B}\widehat{H}) = \mathcal{L}_{d_2}(\mathcal{B}\widehat{H})$, hence $F_{d_1} = F_{d_2}$.*

Proof. Deform the contour of the Laplace integral from direction d_1 to direction d_2 inside the sector (τ_i, τ_{i+1}) . The integrand $e^{-z\xi}g(\xi)$ is holomorphic on the swept region and decays at infinity there, by [lemma 1.9](#); the Cauchy theorem then says that the two integrals agree. Equivalently, there are no singular directions of g between d_1 and d_2 , so no residue is picked up. $\square$

3.3 Stokes matrices and triangularity

Definition 3.7. For $h \notin \mathbb{Q}$, $u \in \mathfrak{h}_{\text{reg}}$ and admissible d , the *Stokes matrix* $S_d(u) \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ is determined by

$$F_{d+\pi}(z) = F_d(z) \cdot S_d(u). \quad (3.10)$$

With J the permutation matrix of the dominance order of $\{e^{hu_1z}, \dots, e^{hu_nz}\}$ along $d + \frac{\pi}{2}$:

Lemma 3.8. $S_d(u)$ and $S_{d+\pi}(u)$ are upper and lower triangular up to conjugation by $1 \otimes J$, and

$$S_d(u)_{kk}^{-1} = S_{d+\pi}(u)_{kk} = 1 \in \text{End}(V), \quad k = 1, \dots, n.$$

Proof. By (3.10), in $\text{Sect}_d \cap \text{Sect}_{d+\pi}$ we have

$$e^{huz} z^{h\delta T} e^{-\pi i h \delta T} S_d(u) e^{-huz} z^{-h\delta T} = \left(F_d z^{-\delta T} e^{-uz} \right)^{-1} \left(F_{d+\pi} (ze^{-\pi i})^{-\delta T} e^{-uz(\cdot)} \right) \longrightarrow 1,$$

where the factor $e^{-\pi i h \delta T}$ accounts for the change of branch of $\log z$ upon rotating the direction by π , and the two factors on the right tend to 1 by (3.9). Write $S_d(u) = \sum_{i,j} (S_d)_{ij} \otimes E_{ij}$ with $(S_d)_{ij} \in \text{End}(V)$. The (i, j) entry of the left hand side is

$$e^{h(u_i - u_j)z} z^{h(e_{ii} - e_{jj})} e^{-\pi i h(e_{ii} - e_{jj})} (S_d)_{ij},$$

and for this to be bounded as $z \rightarrow \infty$ along $d + \frac{\pi}{2}$ we must have $(S_d)_{ij} = 0$ unless $e^{h(u_i - u_j)z} \rightarrow 0$ there, i.e. unless $e^{hu_i z}$ is dominated by $e^{hu_j z}$. This is precisely the triangularity after conjugation by J . For $i = j$ the entry is $(S_d)_{ii}$, so $(S_d)_{ii} = 1$.

Finally, only the diagonal part of $e^{-\pi i h \delta T} S_d(u)$ can be nonzero, since $e^{-\pi i h \delta T}$ is diagonal; hence

$$S_d(u) = e^{\pi i h \delta T} \cdot (\text{unipotent}), \quad (3.11)$$

which is the reason the numerical R -matrix of equation (4.1) is a diagonal half-turn times a unipotent matrix. $\square$

Lemma 3.9. $S_{d+2\pi}(u) = e^{2\pi i \delta T} S_d(u) e^{-2\pi i \delta T}$ for every admissible d .

Proof. Identical to lemma 2.10, with $e^{2\pi i h \delta T}$ in place of $e^{2\pi i [A]}$. $\square$

From now on we choose d so that the dominance order is the identity:

$$\text{Im}(hu_1 e^{id}) > \text{Im}(hu_2 e^{id}) > \dots > \text{Im}(hu_n e^{id}), \quad (3.12)$$

and write $S_{h+}(u) := S_d(u)$ and $S_{h-}(u) := S_{d+\pi}(u)$, which are then upper and lower triangular respectively. As in (2.4) the matrix $S_{h\pm}$ is an ordered product of the Stokes factors, up to the half-turn $e^{\pi i h \delta T}$.

4 The RLL relation

4.1 The numerical R -matrix

Let

$$R = \sum_{i \neq j} E_{ii} \otimes E_{jj} + e^{\pi i h} \sum_{i=1}^n E_{ii} \otimes E_{ii} + (e^{\pi i h} - e^{-\pi i h}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji} \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n). \quad (4.1)$$

Writing $\delta P = \sum_i E_{ii} \otimes E_{ii}$, which commutes with the third sum and is killed by the first, we have the factorization

$$R = e^{\pi i \delta P} R_0, \quad R_0 = \sum_{i \neq j} E_{ii} \otimes E_{jj} + \sum_i E_{ii} \otimes E_{ii} + (e^{\pi i h} - e^{-\pi i h}) \sum_{j < i} E_{ij} \otimes E_{ji}. \quad (4.2)$$

The factor R_0 is unipotent: with $N = \sum_{j < i} E_{ij} \otimes E_{ji}$ one has $N^2 = 0$ and $\delta P N = N \delta P = 0$, so $R_0 = \text{Id} + (q - q^{-1})N$ with $q = e^{\pi i h}$. For $n = 2$ in the basis $(11, 12, 21, 22)$,

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (4.3)$$

Proposition 4.1. *The Stokes matrix of the numerical equation*

$$\frac{dX}{d\omega} = h \left(u + \frac{P}{\omega} \right) X, \quad P = - \sum_{k,l} 1 \otimes E_{kl} \otimes E_{lk}, \quad X \in \mathbb{C}^n \otimes \mathbb{C}^n, \quad (4.4)$$

equals R_0 .

Proof. Let $\{v_i\}$ be the standard basis of $\mathbb{C}^n$. The operator P is the permutation operator $P(v_i \otimes v_j) = -v_j \otimes v_i$, and u acts diagonally with $u(v_i \otimes v_j) = u_j v_i \otimes v_j$. Therefore (4.4) preserves each line $\mathbb{C}(v_i \otimes v_i)$ and each plane $B_{ij} = \text{span}(v_i \otimes v_j, v_j \otimes v_i)$, and we may compute block by block.

The lines. On $\mathbb{C}(v_i \otimes v_i)$ the equation is the scalar equation $X' = hu_i X$, whose Stokes matrix is trivial.

The planes. On B_{ij} with $i \neq j$, ordered as $(v_i \otimes v_j, v_j \otimes v_i)$, the equation is

$$X' = \left[h \begin{pmatrix} u_j & 0 \\ 0 & u_i \end{pmatrix} + \frac{1}{\omega} \begin{pmatrix} 0 & -h \\ -h & 0 \end{pmatrix} \right] X. \quad (4.5)$$

Remove the scalar exponential $e^{\frac{h}{2}(u_i + u_j)\omega}$ and rescale $s = h(u_j - u_i)\omega$; since $\frac{d}{d\omega} = h(u_j - u_i)\frac{d}{ds}$, equation (4.5) becomes

$$\frac{dy}{ds} = \left(\frac{1}{2}\sigma_3 - \frac{h}{s}\sigma_1 \right) y, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.6)$$

Eliminating the second component gives the second order equation

$$y_1'' + \frac{1}{s}y_1' + \left(-\frac{1}{4} - \frac{1}{2s} - \frac{h^2}{s^2} \right) y_1 = 0, \quad (4.7)$$

and the substitution $y_1 = e^{-s/2} s^h f(s)$ reduces (4.7) to Kummer's equation

$$s f'' + (1 + 2h - s) f' - (1 + h) f = 0, \quad (4.8)$$

whose solution regular at $s = 0$ is $f(s) = {}_1F_1(1 + h; 1 + 2h; s) = \sum_{r \geq 0} \frac{(1+h)_r}{(1+2h)_r} \frac{s^r}{r!}$. Replacing s^h by s^{-h} gives the second local exponent, so the two solutions are $y_1 = e^{-s/2} s^{\pm h} f_{\pm}(s)$. Kummer's connection formula [lemma 2.15](#) expresses the coefficient relating them through Gamma functions. In the normalization fixed by (4.6)—that is, comparing the two lateral sums of the same formal series along the two boundary rays of the plane B_{ij} , with the branch of s^h chosen so that s^h is the dominant factor—the surviving lateral coefficient is

$$\frac{2\pi i h}{\Gamma(1+h)\Gamma(1-h)} = 2i \sin(\pi h) = q - q^{-1}, \quad (4.9)$$

the middle equality being Euler's reflection formula $\Gamma(1+h)\Gamma(1-h) = \frac{\pi h}{\sin \pi h}$. So the Stokes matrix of the 2×2 block is $\begin{pmatrix} 1 & q-q^{-1} \\ 0 & 1 \end{pmatrix}$ in the ordering of (4.5).

Assembling the blocks: the diagonal blocks contribute the identity, and each plane contributes a single off-diagonal entry $q - q^{-1}$ in the position corresponding to $j < i$ after the ordering is fixed by (3.12). The result is exactly R_0 . In particular the answer is independent of u and of V . $\square$

Remark 4.2. Two features of proposition 4.1 are worth isolating. First, the numerical equation (4.4) contains no representation of $\mathfrak{gl}_n$, so R is universal: it depends only on n and on h . Second, the u -independence is a cancellation: the gaps $u_j - u_i$ appear in the rescaling of s but cancel in the final coefficient (4.9), because the connection coefficient of Kummer's equation depends only on the exponents $\pm h$ of the residue matrix. This is what allows the same matrix to serve in the comparison of paths for every u .

4.2 The irregular KZ system

Let

$$u^{(1)} = \sum_i 1 \otimes u_i E_{ii} \otimes 1, \quad u^{(2)} = \sum_i 1 \otimes 1 \otimes u_i E_{ii}, \quad (4.10)$$

$$T^{(1)} = \sum_{k,l} e_{kl} \otimes E_{kl} \otimes 1, \quad T^{(2)} = \sum_{k,l} e_{kl} \otimes 1 \otimes E_{kl}, \quad P = - \sum_{k,l} 1 \otimes E_{kl} \otimes E_{lk}, \quad (4.11)$$

all viewed as elements of $U(\mathfrak{gl}_n) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$. We consider the system, for an $\text{End}(V) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ -valued function $Y(z_1, z_2)$,

$$\frac{1}{h} \frac{\partial Y}{\partial z_1} = \left(u^{(1)} + \frac{T^{(1)}}{z_1} + \frac{P}{z_1 - z_2} \right) Y, \quad (4.12)$$

$$\frac{1}{h} \frac{\partial Y}{\partial z_2} = \left(u^{(2)} + \frac{T^{(2)}}{z_2} + \frac{P}{z_2 - z_1} \right) Y. \quad (4.13)$$

This is a Knizhnik–Zamolodchikov type system with an irregular singularity at infinity. The two points z_1 and z_2 are to be thought of as the two places where the auxiliary factor $\mathbb{C}^n$ of (3.2) sits, and the comparison of the asymptotics as the points separate in different orders is what produces the exchange relation.

Lemma 4.3. $[T^{(1)}, T^{(2)}] = [T^{(1)}, P] = [P, T^{(2)}]$, and consequently

$$[T^{(1)} + T^{(2)}, P] = 0, \quad [T^{(1)}, T^{(2)} + P] = 0. \quad (4.14)$$

Proof. Compute on the auxiliary basis. Since $P(v_p \otimes v_q) = -v_q \otimes v_p$ and $E_{ij}v_p = \delta_{jp}v_i$,

$$PT^{(1)}(v_p \otimes v_q \otimes w) = - \sum_{i,j,k} \delta_{kp} \delta_{iq} E_{ki} \otimes E_{ij} \otimes e_{kj} w,$$

hence $PT^{(1)} = - \sum_{i,j,k} E_{ki} \otimes E_{ij} \otimes e_{kj}$. In the same way

$$T^{(1)}P = - \sum_{i,j} E_{ij} \otimes E_{ji} \otimes e_{ii}, \quad T^{(2)}P = - \sum_{i,j,k} E_{ik} \otimes E_{ij} \otimes e_{kj},$$

$$PT^{(2)} = - \sum_{i,j,k} E_{ij} \otimes E_{ik} \otimes e_{ik}, \quad T^{(1)}T^{(2)} = \sum_{i,j,k,l} e_{ij} e_{kl} \otimes E_{ki} \otimes E_{jj}.$$

Comparing the coefficient of $E_{ab} \otimes E_{cd} \otimes e_{rs}$ in $[T^{(1)}, P]$ and in $[P, T^{(2)}]$ gives $\delta_{cd}(\delta_{ra}\delta_{sb} - \delta_{sa}\delta_{rb})$ in both cases, so those two agree. For $[T^{(1)}, T^{(2)}]$ the same computation gives $\delta_{cd}([e_{ar}, e_{ab}] - \delta_{ra}e_{cb})$

at $d = b$ from the first term and $-\delta_{cd}\delta_{ra}\delta_{ab}e_{cb}$ from the second; the second summands cancel against each other by (3.1), leaving $\delta_{cd}[e_{ar}, e_{ab}]$, which by (3.1) is again $\delta_{cd}(\delta_{ra}\delta_{sb} - \delta_{sa}\delta_{rb})$. So all three commutators coincide; call the common value C . Then $[T^{(1)} + T^{(2)}, P] = C - C = 0$ and $[T^{(1)}, T^{(2)} + P] = C - C = 0$, which is (4.14). $\square$

Proposition 4.4. *The system (4.12)–(4.13) is compatible, i.e. $\partial_{z_2}(hA_1) = \partial_{z_1}(hA_2) + [hA_1, hA_2]$ for the coefficient matrices A_1, A_2 of (4.12)–(4.13).*

Proof. Write $d = z_1 - z_2$. The curvature is

$$\partial_{z_2}A_1 - \partial_{z_1}A_2 + [A_1, A_2].$$

The terms involving P contribute $\partial_{z_2}\frac{P}{d} = P/d^2$ and $\partial_{z_1}(-\frac{P}{d}) = P/d^2$, and since $[P, T^{(2)}] = -C$ and $[T^{(1)}, P] = -C$ all contributions proportional to P cancel against each other; those proportional to u cancel because $[u^{(1)} + u^{(2)}, P] = 0$ by (4.14). What remains is h^2 times

$$\frac{C}{z_1z_2} + \frac{C}{z_1d} - \frac{C}{z_2d},$$

which vanishes by the identity $\frac{1}{z_1z_2} = \frac{1}{d}\left(\frac{1}{z_2} - \frac{1}{z_1}\right)$ evaluated with $d = z_1 - z_2$. $\square$

Remark 4.5. The minus sign in P is essential: with P replaced by $P_0 = -P$ one gets $[T^{(1)} + T^{(2)}, P_0] = 2C \neq 0$ and the system is no longer flat. Since the Stokes matrix of (4.4) is unipotent, the sign also determines whether the off-diagonal entry of R_0 sits above or below the diagonal.

4.3 Formal solutions in the three regimes

Lemma 4.6. *For $h \notin \mathbb{Q}$ and fixed $z_2 \neq 0$, equation (4.12) has a unique formal fundamental solution*

$${}_1\widehat{W}(z_1; z_2) = \left(1 + \sum_{m \geq 1} K_m(z_2) z_1^{-m}\right) e^{hz_1 u^{(1)}} z_1^{h\delta T^{(1)}} z_1^{h\delta P}, \quad (4.15)$$

where each $K_m(z_2) = \sum_{i=0}^{m-1} K_{m,i} z_2^i$ is a polynomial of degree $m-1$ in z_2 with coefficients $K_{m,i} \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$.

Proof. For fixed $z_2 \neq 0$ we may expand $\frac{P}{z_1 - z_2} = \sum_{k \geq 0} \frac{P z_2^k}{z_1^{k+1}}$, a convergent expansion for $|z_1| > |z_2|$. Substituting (4.15) into (4.12) and comparing the coefficients of z_1^{-m-1} gives

$$[K_{m+1}(z_2), u^{(1)}] = \sum_{r=0}^m P \cdot K_{m-r}(z_2) z_2^r + K_m \delta P - K_m(z_2) \delta T^{(1)} + \left(\frac{m}{h} + T^{(1)}\right) K_m(z_2). \quad (4.16)$$

The argument of proposition 3.2 applies verbatim, with $u^{(1)}$ in place of u and $T^{(1)}$ in place of T : the off-diagonal entries of K_{m+1} in the second auxiliary factor are determined by division by the constant gaps $u_l - u_k$, and the diagonal ones by the invertibility of $\frac{m+1}{h} \text{Id} + \text{ad}_{e_{kk}}$ on $\text{End}(V)$, which holds since $h \notin \mathbb{Q}$.

The polynomiality is a separate statement and follows from the same recursion: if $K_m(z_2)$ is a polynomial of degree $\leq m-1$, then each term on the right of (4.16) is a polynomial of degree $\leq m-1$ except the sum $\sum_{r=0}^m P K_{m-r}(z_2) z_2^r$, whose r -th summand has degree $\leq (m-r-1) + r = m-1$; hence K_{m+1} is a polynomial of degree $\leq m$. The base case is K_1 , which by (4.16) with $m=0$ equals $P K_0 z_2^{-0} \cdot (\text{gap})^{-1}$ plus constants, a polynomial of degree $0 = m-1+1$; more carefully K_1 is constant, of degree $0 = m-1$ with $m=1$. $\square$

Definition 4.7. Let $\widehat{F}^{(2)}(z_2) \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ be the unique formal solution of

$$\frac{1}{h} \frac{dF}{dz_2} = \left(u^{(2)} + \frac{T^{(2)}}{z_2} \right) F.$$

It is the extension of the $\widehat{F}$ of [proposition 3.2](#) from $\text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ to three tensor factors, sending $\sum_a X_a \otimes Y_a$ to $\sum_a X_a \otimes 1 \otimes Y_a$.

Proposition 4.8. *The product ${}_1\widehat{W}(z_1; z_2)\widehat{F}^{(2)}(z_2)$ is a formal solution of (4.12) taking the form*

$$\left(1 + \sum_{p \geq 0} \widehat{L}_p(z_2) z_1^{-p} \right) e^{hz_2 u^{(2)}} z_2^{h\delta T^{(2)}} e^{hz_1 u^{(1)}} z_1^{h\delta T^{(1)} + h\delta P}, \quad (4.17)$$

with $\widehat{L}_p$ given by the convolution of the K_m with the coefficients of $\widehat{F}^{(2)}$.

Proof. Both factors solve (4.12): ${}_1\widehat{W}$ by construction, and $\widehat{F}^{(2)}$ because $u^{(2)}$ and $T^{(2)}$ act on the third tensor factor only while (4.12) involves the first two, so multiplying a solution on the right by a solution of the $\widehat{F}^{(2)}$ -equation preserves the equation. Writing $\widehat{F}^{(2)} = \widehat{F}_0^{(2)}(1 + \sum_{p \geq 1} F_p^{(2)} z_2^{-p})$ with $\widehat{F}_0^{(2)} = e^{hz_2 u^{(2)}} z_2^{h\delta T^{(2)}}$ and collecting powers of z_1 gives (4.17), the coefficients being the finite convolutions $\widehat{L}_p = \sum_{m+r=p} K_m F_r^{(2)}$; the form of the exponential factors is unchanged under multiplication since they all involve diagonal operators. $\square$

By symmetry, with z_1 and z_2 interchanged, one defines ${}_2\widehat{W}(z_2; z_1)$ and $\widehat{F}^{(1)}(z_1)$, and the analogous products ${}_2\widehat{W}\widehat{F}^{(1)}$ are formal solutions. For the third regime, put $\omega_1 = z_1$ and $\omega_2 = z_2 - z_1$ and define $\widehat{U}(\omega_1; \omega_2)$ and $\widehat{X}^{(2)}(\omega_2)$ so that $\widehat{U}\widehat{X}^{(2)}$ is a formal solution. In ratio coordinates $z_1 = z$, $z_2 = zt$ the common formal solution is

$$\widehat{Y} = \widehat{Q}(z, t) \Theta(z, t), \quad \Theta = e^{hz(u^{(1)} + tu^{(2)})} z^{hD} t^{h\delta T^{(2)}} (t-1)^{h\delta P}, \quad \widehat{Q} = 1 + \sum_{m \geq 1} Q_m(t) z^{-m}. \quad (4.18)$$

Proposition 4.9. *The coefficients $Q_m(t)$ are determined by*

$$[Q_{m+1}, u^{(1)} + tu^{(2)}] = \left(\frac{m}{h} + T^{(1)} + T^{(2)} + P \right) Q_m - Q_m D, \quad D = \delta T^{(1)} + \delta T^{(2)} + \delta P, \quad (4.19)$$

and the only possible poles of $Q_m(t)$ are $t = 0, 1$ and ∞ .

Proof. The z -equation gives (4.19) exactly as in [proposition 3.2](#). For generic t the gaps of $u^{(1)} + tu^{(2)}$ are $u_j - u_i + t(u_l - u_k)$, which vanish for some $t \neq 0, 1$ whenever $u_j - u_i$ and $u_l - u_k$ have opposite signs; so (4.19) alone appears to produce spurious poles. The point is that the second equation removes them. Writing the t -equation as

$$[Q_{m+1}, u^{(2)}] = C(t)Q_m - Q_m E(t) - h^{-1} \partial_t Q_m, \quad C(t) = \frac{T^{(2)}}{t} + \frac{P}{t-1}, \quad E(t) = \frac{\delta T^{(2)}}{t} + \frac{\delta P}{t-1}, \quad (4.20)$$

we see that (4.20) divides only by the fixed gaps $u_l - u_k$ whenever the second auxiliary indices of the entry differ. When they agree but the first indices differ, one uses (4.19) and divides by $u_j - u_i$. The fully diagonal entries require no division. Since the t -dependence of the right hand sides is polynomial away from $t = 0, 1$, an induction on m gives the pole statement.

Finally we must justify that the formal solution of (4.19) also satisfies (4.20), i.e. that the two formal solutions agree. Apply the operator $\partial_t - h(zu^{(2)} + C(t))$ to $\widehat{Y}$. By flatness ([proposition 4.4](#)) the result is again a formal solution of the z -equation (4.19), and its normalized leading coefficient vanishes, because the off-diagonal equation for Q_1 cancels the degree zero term. Formal uniqueness forces the whole defect to vanish. $\square$

4.4 Factorization and connection formulas

For a fixed t the eigenvalues of the irregular term $u^{(1)} + tu^{(2)}$ of (4.12) are $u_i + tu_k$, so its anti-Stokes directions are the arguments of $-(u_i - u_j) - t(u_k - u_l)$. As t varies these rays move, and the interesting ones are those which appear in the three limits $t \rightarrow 0$, $t \rightarrow 1$, $t \rightarrow \infty$.

Theorem 4.10 (Factorization). *Fix $u \in \mathfrak{h}_{\text{reg}}$, $h \notin \mathbb{Q}$ and $d \notin \text{aSR}(hu)$. The sets of rays $\text{aSR}(hu; d)_0$ and $\text{aSR}(hu; d)_1$ cut the universal covering of $\mathbb{C} \setminus \{0, 1\}$ into connected components. For each component D there is a unique holomorphic fundamental solution $Y_{d,D}(z, t)$ of (4.12)–(4.13), defined on the universal covering of $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0, 1\})$, with*

$$Y_{d,D}(z, t) e^{-hz(u^{(1)} + tu^{(2)})} z^{-h\delta T^{(1)}} (tz)^{-h\delta T^{(2)}} (zt - z)^{-h\delta P} \sim \widehat{Q}(z; t), \quad z \rightarrow \infty, \quad -\frac{\pi}{2} < \arg z + d < \frac{\pi}{2}. \quad (4.21)$$

If D_0, D_1, D_2 are the components containing $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$, then

$${}_2\widehat{W}_{d+\pi}(z_2; z_1) F_d^{(1)}(z_1) = Y_{d,D_0}(z, t) = {}_1\widehat{W}_d(z_1; z_2) F_{d+\pi}^{(2)}(z_2) e^{-\pi i \delta P}, \quad (4.22)$$

$${}_1\widehat{W}_d(z_1; z_2) F_d^{(2)}(z_2) e^{-\pi i \delta P} = Y_{d,D_1}(z, t) = U_d(\omega_1; \omega_2) X_{d+\pi}^{(2)}(\omega_2), \quad (4.23)$$

$$U_d(\omega_1; \omega_2) X_d^{(2)}(\omega_2) = Y_{d,D_2}(z, t) = {}_2\widehat{W}_d(z_2; z_1) F_d^{(1)}(z_1). \quad (4.24)$$

Proof. Borel–Laplace transform $\widehat{Q}$ along d . For $|t|$ small, the expansion of $\widehat{Q}$ in powers of t reproduces the series whose Borel sum is ${}_1\widehat{W}_d F_d^{(2)}$, and the direction of summation relevant for the second variable is $d + \arg t$; this gives the middle expression of (4.22) and, by the same computation with the roles exchanged, (4.24). For $|t - 1|$ small, the expansion in powers of $t - 1$ reproduces the series whose Borel sum is $U_d X_{d+\arg(t-1)}^{(2)}$, giving (4.23).

The set of non-admissible directions of $\widehat{Q}$ is

$$\{-\arg(h(u_i - u_j)), -\arg(h(u_i - u_j)t), -\arg(h(u_i - u_j)(t - 1)) : i \neq j\},$$

i.e. the union of the anti-Stokes directions of the three one-point problems. All other directions $\arg(h(u_i - u_j) + ht(u_k - u_l))$ are removable: by proposition 4.9 the coefficients $Q_m(t)$ are smooth at $t = \frac{u_i - u_j}{u_k - u_l} \neq 0, 1$, so the putative singularity is an artifact of the individual blocks and does not affect the sum. Finally, since $d + \arg t$ and d lie in the same connected component D_1 for $t \in D_1$, the canonical solutions coincide by lemma 3.6, which turns the middle expressions into the left hand ones without extra factors. The half-turn factors $e^{\pm \pi i \delta P}$ record the comparison of the two branches of $\log(z_1 - z_2)$ and $\log(z_2 - z_1)$; they belong to the transition and must not be counted twice when R is inserted later. $\square$

Definition 4.11. Following theorem 4.10, introduce the six fundamental solutions

$$\begin{aligned} Y_{012} &= U_d X_d^{(2)} = {}_2\widehat{W}_d F_d^{(1)}, & Y_{021} &= {}_1\widehat{W}_d F_d^{(2)} e^{-\pi i \delta P} = U_d X_{d+\pi}^{(2)}, \\ Y_{201} &= {}_2\widehat{W}_{d+\pi} F_d^{(1)} = {}_1\widehat{W}_d F_{d+\pi}^{(2)} e^{-\pi i \delta P}, & Y_{102} &= {}_2\widehat{W}_d F_{d+\pi}^{(1)} = {}_1\widehat{W}_{d+\pi} F_d^{(2)} e^{\pi i \delta P}, \\ Y_{120} &= {}_1\widehat{W}_{d+\pi} F_{d+\pi}^{(2)} e^{\pi i \delta P} = U_{d+\pi} X_d^{(2)}, & Y_{210} &= U_{d+\pi} X_{d+\pi}^{(2)} = {}_2\widehat{W}_{d+\pi} F_{d+\pi}^{(1)}. \end{aligned} \quad (4.25)$$

The subscripts record which of the three asymptotically separated regions is used on the left and on the right: 0 for $t \rightarrow 0$, 1 for $t \rightarrow 1$ and 2 for $t \rightarrow \infty$, in the order in which the two expressions are written.

Lemma 4.12.

$$\begin{aligned} Y_{021} &= Y_{201} e^{\pi i \delta P} S_d^{(2)}(u)^{-1} e^{-\pi i \delta P}, & Y_{201} &= Y_{210} S_d^{(1)}(u)^{-1}, \\ Y_{012} &= Y_{102} S_d^{(1)}(u)^{-1}, & Y_{102} &= Y_{120} e^{-\pi i \delta P} S_d^{(2)}(u)^{-1} e^{\pi i \delta P}. \end{aligned}$$

Proof. From (4.25), $Y_{021} = {}_1\widehat{W}_d F_d^{(2)} e^{-\pi i \delta P}$ and $Y_{201} = {}_1\widehat{W}_d F_{d+\pi}^{(2)} e^{-\pi i \delta P}$, so

$$Y_{201}^{-1} Y_{021} = e^{\pi i \delta P} (F_{d+\pi}^{(2)})^{-1} F_d^{(2)} e^{-\pi i \delta P} = e^{\pi i \delta P} S_d^{(2)}(u)^{-1} e^{-\pi i \delta P},$$

because $F_{d+\pi}^{(2)} = F_d^{(2)} S_d^{(2)}$ by (3.10). The other three identities are the same computation with the appropriate pair of expressions from (4.25). $\square$

Lemma 4.13. *Under (3.12),*

$$Y_{012} = Y_{021} R_0^{(12)}, \quad Y_{120} = Y_{210} R_0^{(12)},$$

where $R_0^{(12)} = 1 \otimes R_0$ and $R_0 = e^{-\pi i \delta P} R$ is the numerical matrix of proposition 4.1.

Proof. By (4.25), $Y_{012} = U_d X_d^{(2)}$ and $Y_{021} = U_d X_{d+\pi}^{(2)}$, so

$$Y_{021}^{-1} Y_{012} = (X_{d+\pi}^{(2)})^{-1} X_d^{(2)} = S_d^{(2)}(P)^{-1},$$

where $S_d^{(2)}(P)$ is the Stokes matrix of (4.4). That equation acts on the last two tensor factors only, so $S_d^{(2)}(P) = 1 \otimes S_d(P)$ with $S_d(P)$ the Stokes matrix of (4.4), which by proposition 4.1 is R_0 . The second identity is the same computation for the pair Y_{120}, Y_{210} . $\square$

Proposition 4.14. *Under (3.12),*

$$\begin{aligned} Y_{012} &= Y_{210} S_d^{(1)}(u)^{-1} e^{\pi i \delta P} S_d^{(2)}(u)^{-1} e^{-\pi i \delta P} R_0^{(12)}, \\ Y_{012} &= Y_{210} R_0^{(12)} e^{-\pi i \delta P} S_d^{(2)}(u)^{-1} e^{\pi i \delta P} S_d^{(1)}(u)^{-1}. \end{aligned}$$

Proof. Compose the identities of lemma 4.12 and lemma 4.13 along the two chains

$$012 \rightarrow 021 \rightarrow 201 \rightarrow 210, \quad 012 \rightarrow 102 \rightarrow 120 \rightarrow 210,$$

both of which start at Y_{012} and end at Y_{210} . The first chain uses lemma 4.13 once and lemma 4.12 twice; the second uses lemma 4.12 twice and lemma 4.13 once. Since the six matrices are invertible, the two compositions are both equal to $Y_{210}^{-1} Y_{012}$, and hence to each other. $\square$

4.5 The main theorem

Write $S^{(1)} = \sum_{i,j} S_{ij} \otimes E_{ij} \otimes 1$ and $S^{(2)} = \sum_{i,j} S_{ij} \otimes 1 \otimes E_{ij}$ for $S \in \text{End}(V) \otimes \text{End}(\mathbb{C}^n)$, both in $\text{End}(V) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$.

Theorem 4.15 (*RLL relation*). *For $h \notin \mathbb{Q}$, $u \in \mathfrak{h}_{\text{reg}}$ and d satisfying (3.12), the Stokes matrices $S_{h\pm}(u)$ of (3.2) satisfy*

$$R_{12} S_{h\pm}^{(1)} S_{h\pm}^{(2)} = S_{h\pm}^{(2)} S_{h\pm}^{(1)} R_{12}, \quad (4.26)$$

$$R_{12} S_{h+}^{(1)} S_{h-}^{(2)} = S_{h-}^{(2)} S_{h+}^{(1)} R_{12}, \quad (4.27)$$

in $\text{End}(V) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$, where $R_{12} = 1 \otimes R$.

Proof. Equating the two expressions of proposition 4.14 and cancelling the common factor Y_{210} gives

$$S_d^{(1)}(u)^{-1} e^{\pi i \delta P} S_d^{(2)}(u)^{-1} e^{-\pi i \delta P} R_0^{(12)} = R_0^{(12)} e^{-\pi i \delta P} S_d^{(2)}(u)^{-1} e^{\pi i \delta P} S_d^{(1)}(u)^{-1}. \quad (4.28)$$

Taking inverses of both sides and rearranging,

$$S_d^{(2)}(u)e^{\pi i \delta P} S_d^{(1)}(u) R_0^{(12)} e^{\pi i \delta P} = e^{\pi i \delta P} R_0^{(12)} S_d^{(1)}(u) e^{\pi i \delta P} S_d^{(2)}(u). \quad (4.29)$$

Note that R_0 commutes with $e^{\pi i \delta P}$, since δP is diagonal and R_0 preserves the decomposition into E_{ij} -blocks with $i = j$ and $i \neq j$. Multiply (4.29) on the left by $e^{-\pi i \delta T^{(1)}} e^{-\pi i \delta T^{(2)}}$. For $k = 2$ we use

$$e^{-\pi i \delta T^{(1)}} S_d^{(2)}(u) = S_d^{(2)}(u) e^{-\pi i \delta T^{(1)}}, \quad (4.30)$$

which holds because $\delta T^{(1)}$ and $S_d^{(2)}$ act on disjoint tensor factors; for $k = 1$ the analogous factor $e^{-\pi i \delta T^{(1)}}$ must be moved past $S_d^{(1)}(u)$, and there one uses that both are built from the same $\text{End}(V) \otimes \text{End}(\mathbb{C}^n)$ factor, so $e^{-\pi i \delta T} S_d(u) = e^{-\pi i h \delta T} S_d(u)$ and, by (3.11), $S_d(u) e^{\pi i h \delta T}$ is unipotent. Combining the two factors into $e^{-\pi i \delta P} e^{-\pi i \delta P} = \text{Id}$ and using $R = R_0 e^{\pi i \delta P} = e^{\pi i \delta P} R_0$ turns the result into (4.26). The mixed relation (4.27) is obtained from the other two orderings of the six regions in the same way, the only difference being which of S_{h+}, S_{h-} appears on which side. $\square$

Remark 4.16. The content of theorem 4.15 is that the product $S^{(1)} S^{(2)}$ computed in the two orders of the auxiliary factors differs by conjugation with R_{12} . Concretely, if one writes the two auxiliary factors as two ordered lattices of asymptotic regions and follows the boundary of the resulting picture, the six fundamental solutions of definition 4.11 sit at the six corners, the one-point Stokes matrices label the edges where a single asymptotic parameter varies, and R labels the edge where the two points collide. Two paths around the same region give the same result, and the resulting identity is $RLL = LLR$.

5 Representation of $U_q(\mathfrak{gl}_n)$

5.1 The FRT realization

Let $L^\pm = \sum_{i,j} l_{ij}^{(\pm)} \otimes E_{ij} \in U(R) \otimes \text{End}(\mathbb{C}^n)$ with $l_{ij}^{(\pm)} = 0$ for $i > j$, so that L^+ is upper and L^- is lower triangular. The algebra $U(R)$ is generated by the entries $l_{ij}^{(\pm)}$, $1 \leq i \leq j \leq n$, subject to the relations

$$R_{12} L^{\pm(1)} L^{\pm(2)} = L^{\pm(2)} L^{\pm(1)} R_{12}, \quad (5.1)$$

$$R_{12} L^{+(1)} L^{-(2)} = L^{-(2)} L^{+(1)} R_{12}, \quad (5.2)$$

$$l_{ii}^{(+)} l_{ii}^{(-)} = l_{ii}^{(-)} l_{ii}^{(+)} = 1.$$

This is the Faddeev–Reshetikhin–Takhtajan presentation of the quantized enveloping algebra [4]. The two presentations are isomorphic, and an explicit isomorphism $I: U_q(\mathfrak{gl}_n) \rightarrow U(R)$ is given on generators by

$$l_{ii}^{(+)} = q^{-h_i}, \quad l_{i,i+1}^{(+)} = -(q - q^{-1}) q^{-h_i} e_i, \quad l_{i+1,i}^{(-)} = (q - q^{-1}) f_i q^{h_i}, \quad (5.3)$$

see [3]. Here $U_q(\mathfrak{gl}_n)$ is the unital associative algebra generated by $q^{\pm h_i}$, e_j , f_j for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, with relations

$$\begin{aligned} q^{h_i} q^{-h_i} &= q^{-h_i} q^{h_i} = 1, & q^{h_i} e_j q^{-h_i} &= q^{\delta_{ij}} q^{-\delta_{i,j+1}} e_j, & q^{h_i} f_j q^{-h_i} &= q^{-\delta_{ij}} q^{\delta_{i,j+1}} f_j, \\ [e_i, f_j] &= \delta_{ij} \frac{q^{h_i - h_{i+1}} - q^{-h_i + h_{i+1}}}{q - q^{-1}}, & [e_i, e_j] &= [f_i, f_j] = 0 \quad (|i - j| \neq 1), \\ e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0, & f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0 \quad (|i - j| = 1). \end{aligned} \quad (5.4)$$

Example 5.1. Take $n = 2$ and write $L^+ = \begin{pmatrix} k_1 & b \\ 0 & k_2 \end{pmatrix}$ and $L^- = \begin{pmatrix} k_1^{-1} & 0 \\ c & k_2^{-1} \end{pmatrix}$. It is instructive to see the whole set of relations come out of the two matrix identities (5.1)–(5.2), so we record the computation. With R as in (4.3) and $\gamma = q - q^{-1}$, comparing the $(1, 2)$ entries of $R_{12}L_1^+L_2^+ = L_2^+L_1^+R_{12}$ gives $k_1b = qbk_1 + \gamma k_2b$ and γbk_1 from the two sides, which combine to

$$k_1b = q^{-1}bk_1, \quad k_2b = qbk_2;$$

the $(2, 1)$ entries give the transpose relations and the $(1, 1)$, $(2, 2)$ entries are the trivial ones. The mixed relation (5.2) gives, from its $(1, 2)$ entry,

$$k_1c = qck_1, \quad k_2c = q^{-1}ck_2,$$

and, from its $(2, 2)$ entry,

$$bc - cb = \gamma(k_1k_2^{-1} - k_1^{-1}k_2). \quad (5.5)$$

Setting $K_a = k_a^{-1}$, $E = -k_1^{-1}b/\gamma$ and $F = ck_1/\gamma$: the second of the first two relations says K_2 commutes with b , and the first of the second pair says K_1 commutes with c . Hence bc and cb may be brought together in $[E, F]$ and

$$[E, F] = -\frac{1}{\gamma^2}k_1^{-1}(bc - cb)k_1 = -\frac{bc - cb}{\gamma^2} = \frac{K_1K_2^{-1} - K_1^{-1}K_2}{q - q^{-1}},$$

using (5.5). The four relations also reproduce the Cartan relations of (5.4) for $K_a = k_a^{-1}$, and there are no Serre relations to check for $n = 2$. So the single quadratic matrix identity (5.1) encodes both the Cartan action and the ef commutator: a quadratic relation in the $l_{ij}^{(\pm)}$ is equivalent to a linear action of the Cartan plus a commutator of raising and lowering operators.

5.2 The main theorem

Theorem 5.2. For $h \notin \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, the map $S_q(u): U_q(\mathfrak{gl}_n) \rightarrow \text{End}(V)$ with $q = e^{\pi i h}$ defined by

$$e_i \mapsto \frac{S_{h+}(u)_{i,i}^{-1} S_{h+}(u)_{i,i+1}}{q^{-1} - q}, \quad f_i \mapsto \frac{S_{h-}(u)_{i+1,i} S_{h-}(u)_{i,i}^{-1}}{q - q^{-1}}, \quad q^{h_i} \mapsto S_{h+}(u)_{i,i}, \quad (5.6)$$

defines a representation of $U_q(\mathfrak{gl}_n)$ on V .

Proof. By theorem 4.15 the matrices $L^\pm := S_{h\pm}^{(0)}$ satisfy (5.1)–(5.2), and by lemma 3.8 they are upper and lower triangular with invertible diagonal entries whose product is 1. Hence they give a well-defined algebra homomorphism $U(R) \rightarrow \text{End}(V)$, and composing with the isomorphism (5.3) gives (5.6). The three formulas in (5.6) are exactly the images of the corresponding generators under I^{-1} . $\square$

Remark 5.3. The theorem acts on the same vector space V on which the original e_{ij} acted, but an individual Stokes entry is not the original Lie algebra generator: the entries depend on the spectral parameter u , on the chosen branches, and on the normalization. What the theorem says is that certain functions of them satisfy a prescribed algebraic system, and that this system does not see the transcendental origin of the entries. In particular the representation is not obtained by deforming the original one: for $h \rightarrow 0$ the quantum group degenerates to $U(\mathfrak{gl}_n)$ and the Stokes matrices degenerate to the identity, so the correspondence is genuinely transcendental.

Remark 5.4. The point of having an algebraic description of analytic objects is that algebraic methods become available for analytic problems. Two instances indicated in [6, 7] are the following. The description of the Stokes phenomenon in the WKB approximation can be phrased in terms of the canonical basis of Lusztig and Kashiwara, because the Stokes operators of [theorem 5.2](#) are operators on V and can be compared with the crystal structure. And the algebraic characterization of soliton or polynomial solutions of confluent hypergeometric type equations follows from the same identification, since those solutions correspond to specific vectors in V on which the Stokes operators act by scalars.

5.3 The resonant case

The hypothesis $h \notin \mathbb{Q}$ was used only in [proposition 3.2](#), to invert $\frac{m+1}{h}\text{Id} + \text{ad}_{e_{kk}}$. At a rational h_0 that operator may be singular, and then the formal solution is not unique. The remedy is to choose the diagonal coefficients differently.

Proposition 5.5. *For every $h \neq 0$, equation (3.2) has a formal fundamental solution $\hat{F}^o(z) = \hat{H}^o(z)e^{huz}z^{h\delta T}$ with $\hat{H}^o = 1 + \sum_{m \geq 1} H_m z^{-m}$, where the off-diagonal entries of H_{m+1} are given by (3.7) and the diagonal ones are defined by*

$$H_{m+1,kk} = -\frac{h}{m+1} \sum_{j \neq k} e_{kj} H_{m+1,jk}. \quad (5.7)$$

Its coefficients satisfy

$$[e_{aa}, H_{m,kl}] = (\delta_{ak} - \delta_{al}) H_{m,kl}, \quad (5.8)$$

depend holomorphically on $h \in \mathbb{C}^*$, and agree with the coefficients of [proposition 3.2](#) whenever the latter exist.

Proof. Induction on m . The weight statement (5.8) is true for $H_0 = \text{Id}$. Suppose it holds for H_m . Each term on the right of (3.7) has weight $\delta_{ak} - \delta_{al}$: for $e_{kj} H_{m,jl}$ this is the sum $(\delta_{ak} - \delta_{aj}) + (\delta_{aj} - \delta_{al})$, the remaining terms are single factors with the same weight, and division by the scalar gap $u_l - u_k$ changes nothing. So the off-diagonal part of H_{m+1} has the required weights. For the diagonal part, every product $e_{kj} H_{m+1,jk}$ in (5.7) has weight $(\delta_{ak} - \delta_{aj}) + (\delta_{aj} - \delta_{ak}) = 0$, so $H_{m+1,kk}$ commutes with every e_{aa} , in particular with e_{kk} ; hence the left hand side of (3.8), namely $(\frac{m+1}{h}\text{Id} + \text{ad}_{e_{kk}}) H_{m+1,kk}$, reduces to the scalar $\frac{m+1}{h} H_{m+1,kk}$, and (3.8) becomes exactly (5.7). So (3.8) is satisfied and, with it, the formal differential equation. The off-diagonal equations hold by construction.

For the holomorphy: the only divisions in the construction are by h and by positive integers, and the spectral gaps $u_l - u_k$ are fixed nonzero constants, so every coefficient is a polynomial in h and h^{-1} . In particular there is no division by a resonant scalar of the form $m + hr$. Where the nonresonant recursion of [proposition 3.2](#) also applies, its solution is unique, so the two agree there; being holomorphic in h on $\mathbb{C}^*$, they agree wherever both are defined. $\square$

Remark 5.6. The weight condition (5.8) is what replaces the missing inverse: it reduces the diagonal recursion to division by the positive integer $m + 1$. This is the precise sense in which the representation theory rescues the analysis at a resonant parameter.

Proposition 5.7. *Let $h = h_0 \in \mathbb{Q}$. Choose $X_{m,k} \in \text{End}(V)$ with $h_0[e_{kk}, X_{m,k}] = -mX_{m,k}$, $m > 0$, and form, for scalar parameters $c_{m,k}$,*

$$U(z; c) = 1 + \sum_{m,k} c_{m,k} X_{m,k} \otimes E_{kk} z^{-m}, \quad D(c) = 1 + \sum_{m,k} c_{m,k} X_{m,k} \otimes E_{kk}. \quad (5.9)$$

There are finitely many summands once bases of the relevant kernels are chosen. Then $\hat{H}^o(z)U(z; c)e^{h_0uz}z^{h_0\delta T}$ is again a formal solution of (3.2), and

$$U(z; c)e^{h_0uz}z^{h_0\delta T} = e^{h_0uz}z^{h_0\delta T}D(c). \quad (5.10)$$

Consequently the Stokes matrices of all these formal solutions are conjugate:

$$S(u; c) = D(c)S^o(u)D(c)^{-1}, \quad (5.11)$$

where $S^o(u)$ are the Stokes matrices of the distinguished solution of [proposition 5.5](#).

Proof. Differentiating U termwise, $U' = -\sum m c_{m,k} X_{m,k} \otimes E_{kk} z^{-m-1}$, while $\frac{h_0}{z}[\delta T, U] = \frac{h_0}{z} \sum c_{m,k} [e_{kk}, X_{m,k}] \otimes E_{kk} z^{-m} = \sum c_{m,k} m X_{m,k} \otimes E_{kk} z^{-m-1}$ by the defining relation of $X_{m,k}$. Adding, $U' + \frac{h_0}{z}[\delta T, U] = 0$, which is the condition for $\hat{H}^o U$ to solve the equation for $\hat{H}$; this proves the first claim. For (5.10), use again $z^{-m} X_{m,k} = X_{m,k}(z^{-m} \cdot)$ together with $e^{h_0uz}z^{h_0\delta T}(X_{m,k} \otimes E_{kk}) = (X_{m,k} \otimes E_{kk})e^{h_0uz}z^{h_0\delta T}z^{-m}$, which is the weight relation written in exponential form. Finally (5.11) follows from $F(z; c) = F^o(z)D(c)$ on each Stokes sector, since the Stokes matrices are the transition matrices between these solutions and $D(c)$ is constant. Invertibility of $D(c)$ holds because within each auxiliary block the nonconstant part of $D(c)$ lowers the eigenvalue of $h_0 e_{kk}$ by a positive integer, so some power of it vanishes on the finite spectrum of V . $\square$

Theorem 5.8. For any $h_0 \in \mathbb{Q}$ and $u \in \mathfrak{h}_{\text{reg}}$, the map (5.6) with $q_0 = e^{\pi i h_0}$ a root of unity defines a representation of $U_{q_0}(\mathfrak{gl}_n)$.

Proof. The distinguished Stokes matrices $S_{h_0\pm}^o(u)$ are the continuous extension of $S_{h\pm}(u)$ from $h \in \mathbb{C} \setminus \mathbb{Q}$ to h_0 , which exists by [proposition 5.5](#) at the level of formal solutions and hence, by parameter-dependent summation, at the level of canonical solutions and their transitions. The relations (4.26)–(4.27) are polynomial in h and in the matrix entries, so they survive the limit, and [theorem 5.2](#) applies at h_0 . The only caveat is that the generator formulas (5.6) involve division by $q_0 - q_0^{-1}$, which fails at $q_0 = \pm 1$; there one uses limiting generators or a specified specialization of the quantum algebra. The polynomial FRT relations (5.1)–(5.2) remain meaningful at every root of unity. $\square$

Example 5.9 (A vanishing Stokes entry). Let $h = \frac{m}{p}$ be a positive rational and let Λ^o be a Gelfand–Tsetlin pattern with $\lambda_1^{(1)}(\Lambda^o) - \lambda_2^{(2)}(\Lambda^o) = p - 1$, so that $e_{12}(\xi_{\Lambda^o}) = c \xi_{\Lambda'}$ with $\lambda_1^{(1)}(\Lambda') - \lambda_2^{(2)}(\Lambda') = p$. Then the $(1, 2)$ entry of the Stokes matrix acts on ξ_{Λ^o} by a multiple of $1/\Gamma(1 + h(\lambda_1^{(1)}(\Lambda') - \lambda_2^{(2)}(\Lambda'))) = 1/\Gamma(1 - m) = 0$, so that entry vanishes identically. This shows that the special-function formula for a Stokes matrix can be singular at a resonant parameter while the Stokes entry itself is finite, or even zero: the singularity belongs to the intermediate basis in which the computation was performed, not to the answer.

5.4 The resonant condition for $\mathfrak{gl}_n$

It is worth recording what resonance means concretely, since the general statement in [proposition 3.2](#) is abstract.

Definition 5.10. Let V be a finite-dimensional $\mathfrak{gl}_n$ -module and $h \in \mathbb{C} \setminus \{0\}$. The system (3.2) is *resonant* if there are $k \in \{1, \dots, n\}$ and $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{h} \text{Id} + \text{ad}_{e_{kk}}$ fails to be invertible on $\text{End}(V)$, i.e. if $-\frac{m}{h}$ is an eigenvalue of $\text{ad}_{e_{kk}}$.

Proposition 5.11. Let V have weights $\lambda, \mu, \dots$ (with respect to the e_{kk}). Then

$$(3.2) \text{ is resonant} \iff \exists k, \exists \text{ weights } \lambda, \mu \text{ of } V, \exists m \in \mathbb{Z}_{>0} : \frac{m}{h} = \mu_k - \lambda_k. \quad (5.12)$$

In particular, if $h \notin \mathbb{Q}$ there is no resonance; if $h \in \mathbb{Q}$ resonance occurs if and only if the corresponding integer is realized as a difference of weights of V .

Proof. The eigenvalues of $\text{ad}_{e_{kk}}$ on $\text{End}(V)$ are the differences $\mu_k - \lambda_k$ of weights, because $\text{End}(V) \cong V \otimes V^*$ and e_{kk} acts on V^* by $-\lambda_k$. So $\frac{m}{h} + \text{ad}_{e_{kk}}$ is singular exactly when $\frac{m}{h} = -(\mu_k - \lambda_k)$ for some pair of weights, which is (5.12). $\square$

Example 5.12. For $V = \mathbb{C}^2$ the dual natural $\mathfrak{gl}_2$ -module with weights $(1, 0)$ under the convention of example 3.1, the weight differences of V are $0, \pm 1$. Hence (3.2) is resonant exactly when $\frac{1}{h} \in \mathbb{Z}$, i.e. $h \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{-1, -\frac{1}{2}, \dots\}$. For such h the operator $\frac{1}{h}\text{Id} + \text{ad}_{e_{11}}$ has a kernel and the formal solution of proposition 3.2 is not unique; the distinguished solution of proposition 5.5 is the one that survives, with the diagonal coefficient determined by (5.7) instead of by an inverse.

Remark 5.13. A rational h is therefore not automatically resonant: everything depends on which weight differences are realized in V . For V one dimensional all weights are equal, all the $\text{ad}_{e_{kk}}$ vanish on $\text{End}(V)$, and there is no resonance for any $h \neq 0$. For the natural module of $\mathfrak{gl}_n$ the weight differences are $0, \pm 1$, so resonance is the condition $h \in \mathbb{Q}$ with numerator ± 1 ; for a module with a large highest weight more rational values of h become resonant.

5.5 The rank two computation in detail

The rank two case can be followed all the way to an explicit formula, and it shows exactly how the Gelfand–Tsetlin labels of V enter the answer. Let V be a finite-dimensional $\mathfrak{gl}_2$ -module of highest weight $\lambda = (\lambda_1^{(2)}, \lambda_2^{(2)})$ with $\lambda_1^{(2)} - \lambda_2^{(2)} \in \mathbb{Z}_{\geq 0}$, and let $\{\xi_\Lambda\}$ be a Gelfand–Tsetlin basis, indexed by the patterns $\Lambda = \{\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}\}$ satisfying the interlacing conditions

$$\lambda_1^{(2)} - \lambda_1^{(1)} \in \mathbb{Z}_{\geq 0}, \quad \lambda_1^{(1)} - \lambda_2^{(2)} \in \mathbb{Z}_{\geq 0}. \quad (5.13)$$

Put

$$\ell_i^{(j)}(\xi_\Lambda) = \lambda_i^{(j)}(\Lambda) - i + 1, \quad 1 \leq i \leq j \leq 2; \quad (5.14)$$

each $\ell_i^{(j)}$ is a diagonal operator on V in this basis.

After normalizing $u_1 = 0, u_2 = 1$ by an affine change of variable, consider

$$\frac{dF}{dz} = h \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} z^{-1} \right] F. \quad (5.15)$$

This is (3.2) for $n = 2$, and the diagonal of $z^{h\delta T}$ is $z^{\frac{h}{2\pi i} T_d}$ with $T_d = \text{diag}(\ell_1^{(2)} + 1, \ell_2^{(2)} + 1)$, because $h\delta T = \frac{h}{2\pi i} \cdot 2\pi i \delta T$ and $2\pi i \delta T$ has diagonal entries built from the $\ell_i^{(j)}$.

Proposition 5.14. Equation (5.15) has the fundamental solution

$$F(z) = \begin{pmatrix} e_{12}Y_{11} & e_{12}Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} z^{\frac{h}{2\pi i} T_d}, \quad (5.16)$$

where, as elements of $\text{End}(V)$,

$$Y_{11} = \frac{{}_1F_1(\alpha_{11}; \beta_{11}; hz)}{\ell_1^{(2)} - \ell_1^{(1)}}, \quad Y_{12} = \frac{{}_1F_1(\alpha_{12}; \beta_{12}; hz)}{\ell_2^{(2)} - \ell_1^{(1)}}, \quad Y_{2j} = {}_1F_1(\alpha_{2j}; \beta_{2j}; hz), \quad (5.17)$$

and the parameters are the diagonal operators

$$\begin{aligned}
 \alpha_{11} &= h(\ell_1^{(2)} - \ell_1^{(1)}), & \beta_{11} &= 1 + h(\ell_1^{(2)} - \ell_2^{(2)}), \\
 \alpha_{12} &= h(\ell_2^{(2)} - \ell_1^{(1)}), & \beta_{12} &= 1 + h(\ell_2^{(2)} - \ell_1^{(1)}), \\
 \alpha_{21} &= 1 + h(\ell_1^{(2)} - \ell_1^{(1)}), & \beta_{21} &= 1 + h(\ell_2^{(2)} - \ell_2^{(2)}), \\
 \alpha_{22} &= 1 + h(\ell_2^{(2)} - \ell_1^{(1)}), & \beta_{22} &= 1 + h(\ell_2^{(2)} - \ell_1^{(1)}).
 \end{aligned} \tag{5.18}$$

Here ${}_1F_1$ is applied to a diagonal operator entrywise.

Sketch. The system (5.15) is the confluent hypergeometric system for the $\mathfrak{gl}_2$ case; the entries (5.17) are obtained by writing the Gelfand–Tsetlin decomposition of F , which reduces the operator equation to the scalar Kummer equation (4.8) on each matrix entry. The appearance of the denominators $\ell_1^{(2)} - \ell_1^{(1)}$ and $\ell_2^{(2)} - \ell_1^{(1)}$ in Y_{11} , Y_{12} is the same division by spectral gaps as in (1.6), now with the gaps of the Gelfand–Tsetlin labels; the interlacing (5.13) guarantees they are nonzero generically. The exponents α, β are the weight differences of (5.18). $\square$

Comparing (5.17) with the prescribed asymptotics on the two Stokes sectors using lemma 2.15 gives the rank two Stokes matrix in closed form. Writing U_0, U_1 for the two transition matrices and using $F = F_0 U_0$ on Sect_0 and $F = F_1 U_1$ on Sect_1 , one has $S_{h+}(u)U_0 = e^{\delta T/2}U_1$, and substituting the Gamma-function coefficients produces

$$S_{h+}(u) = \begin{pmatrix} e^{\frac{h}{2}e_{11}} & -\frac{2\pi i(-h)^{(\cdot)}e^{\frac{h}{2}e_{11}}}{\Gamma(1+h(\ell_1^{(1)}-\ell_1^{(1)}-1))\Gamma(1+h(\ell_1^{(1)}-\ell_1^{(2)}-1))}he_{12} \\ 0 & e^{\frac{h}{2}e_{22}} \end{pmatrix}. \tag{5.19}$$

The precise exponents in the numerator depend on the branch conventions; what matters is the structure, which is exactly that of (3.11): a diagonal half-turn $e^{\frac{h}{2}\delta T}$ times a unipotent upper triangular factor, whose off-diagonal entry is a product of reciprocal Gamma functions of the resonance parameters and the raising operator e_{12} .

Remark 5.15. Formula (5.19) was derived for $h \notin \mathbb{Q}$, because $\beta_{11} = 1 + h(\ell_1^{(2)} - \ell_2^{(2)})$ can take non-positive integer values on part of the spectrum, where ${}_1F_1$ is undefined. It is nevertheless valid for all h , by proposition 5.5 at the level of formal solutions and by continuation at the level of Stokes matrices. At a resonant h the apparently singular factors $\Gamma(1 + \cdot)$ in the denominator are infinite, so the corresponding Stokes entry is finite or zero: the singularity is an artifact of the intermediate basis. Example 5.9 is the extreme case, where the entry vanishes identically.

Acknowledgments

The authors are deeply grateful to Xiaomeng Xu, for introducing this interesting topic and for invaluable guidance, constant encouragement, and insightful feedback throughout the writing.

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