Topology

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Abstract

This note is based on Topology, taught by Jiajun Wang in Fall 2025 at Peking University. Textbooks include Lecture Notes on Basic Topology by Chengye You and Basic Topology by M. A. Armstrong.

Contents

1 Topological Spaces and Continuous Maps

1.1 Topological Spaces

The continuity of maps are important in topology. First recall the continuity of functions and let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$. We say that f is continuous at x_0 if and only if

- for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
- any preimage of a neighborhood of $f(x_0)$ contains some neighborhood of x_0.
- any preimage of an open set containing $f(x_0)$ contains an open set containing x_0.
- any preimage of an open set is open.

Definition 1.1. Let X be a nonempty set. A **topology** on X is a collection τ of subsets of X such that

1. $X \in \tau$ and $\varnothing \in \tau$.
2. Any union of elements of τ is in τ.
3. Any finite intersection of elements of τ is in τ.

3′. Any two elements of τ have their intersection in τ.

A **topological space** is the pair (X, τ) and the elements of τ are called **open sets**.

Example 1.2. 2^X is a topology on X, called the **discrete topology**. $\{X, \varnothing\}$ is a topology on X, called the **trivial topology**.

Let $X = \{a, b, c\}$. Then $\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$ is a topology on X.

Let X be an infinite set and $\tau_f = \{A^c | A \text{ is a finite subset of } X\} \cup \{\varnothing\}$ is a topology on X, called the **finite complement topology**.

Let X be an uncountable infinite set and $\tau_c = \{A^c | A \text{ is a countable subset of } X\} \cup \{\varnothing\}$ is a topology on X, called the **countable complement topology**.

Let $X = \mathbb{R}$ and $\tau_e = \{U | U \text{ is a union of open intervals in } \mathbb{R}\}$ is a topology on X, called the **Euclidean topology** and we write $\mathbb{E}^1 = (\mathbb{R}, \tau_e)$.

We can compare two topologies on the same set. If τ_1 and τ_2 are two topologies on X and $\tau_1 \subseteq \tau_2$, then we say that τ_2 is **finer** or **stronger** than τ_1.

Proposition 1.3. *τ_f is weaker than τ_c and τ_e and we can not compare τ_c and τ_e.*

Proposition 1.4. *Let X be a topological space. Then U is an open set if and only if $\forall x \in U$, there exists an open set O_X such that $x \in O_X \subseteq U$.*

Proof. The necessity is trivial. For the sufficiency, let $U = \bigcup_{x \in U} O_X$. Then U is open. □

Definition 1.5. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geqslant 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.

3. $d(x,z) \leqslant d(x,y) + d(y,z), \forall x,y,z \in X$.

A **metric space** is the pair (X, d).

Let $x_0 \in X$ and $\varepsilon > 0$. A **ball neighborhood** of x_0 with radius ε is the set

$$B(x_0, \varepsilon) = \{x \in X | d(x, x_0) < \varepsilon\}$$

. Denote that $\tau_d = \{U | U \text{ is a union of ball neighborhoods}\}$.

Lemma 1.6. *Any intersection of two ball neighborhoods of X is a union of ball neighborhoods.*

Proof. Let $U = B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. $\forall x \in U$, we have $\varepsilon_i - d(x, x_i) > 0 (i = 1, 2)$. Denote $\varepsilon_x = \min_{i=1,2}\{\varepsilon_i - d(x, x_i)\}$. Then $B(x, \varepsilon_x) \subseteq U$. Hence

$$U = \bigcup_{x \in U} B(x, \varepsilon_x)$$

□

Proposition 1.7. *τ_d is a topology on X.*

Proof. It's easy to see that τ_d satisfies the first two conditions of a topology. For the third condition, let $U, U' \in \tau_d$ and $U = \bigcup_\alpha B(x_\alpha, \varepsilon_\alpha), U' = \bigcup_\beta B(x'_\beta, \varepsilon'_\beta)$. Then

$$U \cap U' = \bigcup_{\alpha,\beta} (B(x_\alpha, \varepsilon_\alpha) \cap B(x'_\beta, \varepsilon'_\beta))$$

is a union of ball neighborhoods, hence $U \cap U' \in \tau_d$. By Lemma 1.6 and condition 2 of a topology, we have $U \cap U' \in \tau_d$. □

τ_d is the topology induced by the metric d and is called the **metric topology**. If $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$, then τ_d is the discrete topology.

Definition 1.8. Let $Y \subseteq X$. The **subspace topology** or **induced topology** on Y induced by τ is $\tau_Y = \{O \cap Y | O \in \tau_X\}$.

Definition 1.9. A is a subset of a topological space X and $x \in A$. x is an **interior point** of A if there exists an open set U such that $x \in U \subseteq A$. U is the **neighborhood** of x. The **interior** of A, denoted by $\mathring{A}$ (or A°), is the set of all interior points of A.

Proposition 1.10. $\mathring{A} = A \iff A$ *is open.*

Proof. If A is open, then $\forall x \in A$, $x \in A \subseteq A$. Then x is an interior point of A and $A \subseteq \mathring{A}$. Hence $A = \mathring{A}$. By Proposition 1.4, we have the converse. □

Proposition 1.11. *$\mathring{A}$ is the union of all the open sets in A, hence the largest open set contained in A.*

Proof. Let $\{U_\alpha\}$ is the set of all open sets contained in A. $\forall x \in \mathring{A}$, there exists an open set U such that $x \in U \subseteq A$. Then $A \subseteq \bigcup U_\alpha$. Conversely, $\forall x \in \bigcup U_\alpha$, there exists an open set U_α such that $x \in U_\alpha \subseteq A$. Then $x \in \mathring{A}$. Hence $\mathring{A} = \bigcup U_\alpha$. □

Corollary 1.12. $(\mathring{A})^\circ = \mathring{A}$.

Proposition 1.13.
- *If $A \subseteq B$, then $\mathring{A} \subseteq \mathring{B}$.*
- $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$.
- $(A \cup B)^\circ \supseteq \mathring{A} \cup \mathring{B}$.

Proof. The first one is trivial. For the second one, $\mathring{A} \cap \mathring{B} \subseteq A \cap B$. Hence $\mathring{A} \cap \mathring{B} \subseteq (A \cap B)^\circ$. Conversely, by the first proposition, $(A \cap B)^\circ \subseteq \mathring{A}$ and $(A \cap B)^\circ \subseteq \mathring{B}$. Hence $(A \cap B)^\circ \subseteq \mathring{A} \cap \mathring{B}$. Therefore, $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$. For the third one, $\mathring{A} \cup \mathring{B}$ is an open subset of $(A \cup B)$. By Proposition 1.11, $\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ$. □

Definition 1.14. A subset A of a topological space X is **closed** if its complement A^c is open.

Proposition 1.15.
- *X and $\varnothing$ are closed.*
- *Any intersection of closed sets is closed.*
- *Any finite union of closed sets is closed.*

Definition 1.16. Let A be a subset of a topological space X and $x \in X$. x is a **limit point** (or **accumulation point**) of A if every neighborhood of x contains a point of A different from x. The **derived set** of A, denoted by A', is the set of all limit points of A. The **closure** of A, denoted by $\overline{A}$, is the set of all limit points of A together with the points of A.

For Euclidean topology, every point in $\mathbb{R}$ is a limit point of $\mathbb{R}$ and no point in $\mathbb{R}^2$ is a limit point of $\{(x, y)|x, y \in \mathbb{Z}\}$.

Proposition 1.17. *A is closed if and only if $\overline{A} = A$.*

Proof. If A is closed, then A^c is open. $\forall x \notin A$, $x \in A^c$. Then there exists an open set U such that $x \in O \subseteq A^c$ and $O \cap A = \varnothing$. Hence x is not a limit point of A and $\overline{A} = A$. Conversely, $\forall x \in A^c$, there exists an open set O such that $x \in O$ and $O \cap A = \varnothing$. Hence $x \in O \subseteq A^c$ and by Proposition 1.4, A^c is open. Therefore, A is closed. □

Example 1.18. For finite complement topology, only infinite sets have limit points and the derived set of $\mathbb{Q}$ is $\mathbb{R}$. For countable complement topology, only uncountable sets have limit points and the derived set of $\mathbb{Q}$ is $\varnothing$.

Proposition 1.19.
- *$\overline{A}$ is the intersection of all closed sets containing A, hence the smallest closed set containing A.*
- *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Proof. The second and the fourth one are trivial.

For the first one, if $x \notin \overline{A}$, then there exists an open set U such that $x \in U \subseteq A^c$. $\overline{A}^c$ is the union of all open sets not containing A and $\overline{A}$ is the intersection of all closed sets containing A.

For the third one, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. By $A \cup B \subseteq \overline{A} \cup \overline{B}$, the second proposition and Proposition 1.17, we have $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. □

Proposition 1.20. *A, B are two subsets of a topological space X and $A^c = B$. Then $\overline{A}^c = \mathring{B}$.*

Proof. $x \in \overline{A}^c \Leftrightarrow$ there exists a neighborhood U of x such that $U \cap A = \varnothing \Leftrightarrow U \subseteq B \Leftrightarrow x \in \mathring{B}$. □

Definition 1.21. A is a subset of a topological space X and A is **dense** if $\overline{A} = X$. X is a **separable space** if X contains countable dense subsets.

Proposition 1.22. *A is dense if and only if every nonempty open set contains a point of A.*

1.2 Topological Basis

Define $\overline{\mathscr{B}} = \{U | U \text{ is a union of elements of } \mathscr{B}\}$.

Definition 1.23. A collection $\mathscr{B}$ of subsets of a set X is a **basis** for a topology on X if $\overline{\mathscr{B}}$ is a topology on X. $\mathscr{B}$ is a **topological basis** of the topological space (X, τ) if $\overline{\mathscr{B}} = \tau$.

All the open intervals form a topological basis of $\mathbb{E}^1$. All the intervals with rational end points form a topological basis of $\mathbb{E}^1$.

Definition 1.24. X is **second countable** if X has a countable topological basis.

Definition 1.25. Two topological basis are **equivalent** if they generate the same topology.

Example 1.26. τ_f and τ_c have no countable topological basis.

Theorem 1.27. *$\mathscr{B}$ is a basis for a topology on X if and only if*

1. *$\bigcup_{B \in \mathscr{B}} B = X$;*
2. *If $B_1, B_2 \in \mathscr{B}$, then $B_1 \cap B_2 \in \overline{\mathscr{B}}$ (i.e. $\forall x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$).*

Proof. The necessity is trivial. For the sufficiency, it satisfies the first two conditions of a topology. For the third condition, let $U, U' \in \overline{\mathscr{B}}$ and $U = \bigcup_\alpha B_\alpha, U' = \bigcup_\beta B'_\beta$. Then $U \cap U' = \bigcup_{\alpha,\beta} (B_\alpha \cap B'_\beta) \in \overline{\mathscr{B}}$. □

For every collection $\mathscr{C}$ of subsets of X, let $B = \{\bigcap_{i=1}^n U_i | n \in \mathbb{N}, U_i \in \{X\} \cup \mathscr{C}\}$, then B satisfies the two conditions of the theorem and hence is a basis for a topology on X.

1.3 Continuous Maps and Homeomorphisms

Definition 1.28. Let X and Y be two topological spaces. A map $f : X \to Y$ is **continuous** at x if for every neighborhood V of $f(x)$ in Y, $f^{-1}(V)$ is a neighborhood of x in X. f is **continuous** if f is continuous at every point of X.

f is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous.

Example 1.29.

1. The **identity map** $id : X \to X$ is continuous. The **inclusion map** $i : A \to X$ defined by $i(x) = x, \forall x \in A$ is continuous.

2. The **constant map** $f : X \to Y$ defined by $f(x) = y_0, \forall x \in X$ is continuous.

3. If X has the discrete topology, then every map $f : X \to Y$ is continuous.

4. If Y has the trivial topology, then every map $f : X \to Y$ is continuous.

5. Let $f : (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_c)$ be defined by $f(x) = x$. f is discontinuous at every point, but f^{-1} is continuous.

Proposition 1.30 gives another definition of continuous maps.

Proposition 1.30. *$f : X \to Y$ is continuous if and only if for every open set V in Y, $f^{-1}(V)$ is open in X.*

Proof. If f is continuous, then for every open set V in Y, $\forall x \in f^{-1}(V)$, there exists a neighborhood U of x such that $f(U) \subseteq V$. By Proposition 1.4, $f^{-1}(V)$ is open in X. Conversely, let V be a neighborhood of $f(x)$. Then there exists an open set O such that $f(x) \in O \subseteq V$. By the assumption, $f^{-1}(O)$ is open and contains x. Hence $f^{-1}(V)$ is a neighborhood of x. □

Proposition 1.31. *Let $f : X \to Y$ and $A \subseteq X$. Denote $f_A = f|A : A \to Y$.*

1. *If f is continuous at x, then f_A is continuous at x.*

2. *If A is a neighborhood of x, then when f_A is continuous at x, f is continuous at x.*

Proof. The first one is trivial. For the second one, let V be a neighborhood of $f(x)$. Then there exists an open set $U_A \subseteq A$ such that $x \in U_A \subseteq f_A^{-1}(V) = A \cap f^{-1}(V)$. Let $U_A = U \cap A$, where U is an open set in X, then $U \cap \mathring{A}$ is an open set in X and contains x. □

Proposition 1.32. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps. If both f and g are continuous, then $g \circ f$ is continuous.*

Proof. For open set $W \subseteq Z$, $V = g^{-1}(W)$ is open in Y and $U = f^{-1}(V)$ is open in X. Hence $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = U$ is open in X. □

Proposition 1.33. *The following statements are equivalent:*

1. *$f : X \to Y$ is continuous;*

2. *Let $\mathscr{B}$ be a topological basis for the topology on Y. Then for every $B \in \mathscr{B}$, $f^{-1}(B)$ is open in X;*

3. *$\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$;*

4. *$\forall B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

5. *For every closed set in Y, its preimage is closed in X.*

Proof. $1 \Rightarrow 2$: Trivial.
$3 \Rightarrow 4$: $f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$ implies $\overline{f^{-1}(B)} \subseteq f^{-1}(f(\overline{f^{-1}(B)})) \subseteq f^{-1}(\overline{B})$. □

Example 1.34. Let $f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ be a map from $S^2\backslash\{(0,0,1)\}$ to $\mathbb{R}^2$. f is bijection $g(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$ is the inverse of f. Both f and g are continuous. Hence f is a homeomorphism between $S^2 - \{(0,0,1)\}$ and $\mathbb{R}^2$.

Definition 1.35. $f : X \to Y$ is an **embedding** if f is a homeomorphism onto its image $f(X)$, where $f(X)$ has the subspace topology induced by Y.

Example 1.36. Let $f : [0,1) \to S^1$ be defined by $f(x) = e^{2\pi ix}$. f is a bijection, but not a homeomorphism since f^{-1} is not continuous, i.e. f is not a embedding.

Definition 1.37. $\mathscr{C} \subseteq 2^X$ is a **cover** of X if $\bigcup_{C\in\mathscr{C}} C = X$. If every element of $\mathscr{C}$ is open (closed), then $\mathscr{C}$ is an **open (closed) cover** of X. $\mathscr{C}$ is a **finite cover** if it contains a finite number of elements.

Theorem 1.38 (Glueing Lemma). *Let $\{A_1, A_2, \cdots, A_n\}$ be a finite and closed cover of X. If $f : X \to Y$ restricted to A_i is continuous for every i, then f is continuous.*

Proof. It suffices to show that the preimage of every closed set B in Y is closed in X.

$$f^{-1}(B) = \bigcup_{i=1}^{n} \left(f_i^{-1}(B) \cap A_i\right) = \bigcup_{i=1}^{n} (f_{A_i}^{-1}(B))$$

□

Definition 1.39. A topological space is called a **disc** if it is homeomorphic to $D^2 = \{(x,y) \in \mathbb{E}^2 | x^2+y^2 \leqslant 1\} \in \mathbb{E}^2$. If $h : A \to D^2$ is a homeomorphism, then $h^{-1}(S^1)$ is called a **boundary** of A, denoted by ∂A.

Proposition 1.40. *Let A be a disc and $f : \partial A \to \partial A$ is a homeomorphism. There exists a homeomorphism $F : A \to A$ such that $F|_{\partial A} = f$.*

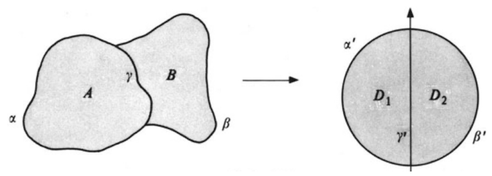
Proof. Choose a homeomorphism $h : A \to D$. Actually, we can show that $h : \partial A \to \partial D$, but we don't give the proof here. Extend $hfh^{-1} : \partial D \to \partial D$ to

$$g(x) = \begin{cases} 0 & x = 0 \\ \|x\| hfh^{-1}\left(\frac{x}{\|x\|}\right) & x \neq 0 \end{cases}.$$

Then $h^{-1}gh$ is the required homeomorphism. □

Proposition 1.41. *Let A and B be two discs and $A \cap B$ is a boundary of both A and B. Then $A \cup B$ is a disc.*

Proof. Let γ denote the arc $A \cap B$, use α, β for the complementary arcs in the boundaries of A, B.



The y axis divides D, as the union of two discs D_1 and D_2. We label three arcs as shown. Both α and α' are homeomorphic to $[0,1]$, so we can find a homeomorphism from α to α'. First extend this over γ, to give a homeomorphism from $\alpha \cup \gamma$ to $\alpha' \cup \gamma'$. By Proposition 1.40, we can extend this to a homeomorphism from A to D_1. Similar to the proof of Proposition 1.40, we can extend the homeomorphism over β, so that β goes to β'. The result is a homeomorphism from $A \cup B$ to D. □

1.4 Product Spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $X \times Y = \{(x, y)|x \in X, y \in Y\}$. We hope the two projections $p_X(x, y) = x$ and $p_Y(x, y) = y$ are continuous. For every $U \in \tau_X$ and $V \in \tau_Y$, $p_X^{-1}(U) = U \times Y$ and $p_Y^{-1}(V) = X \times V$ should be open in $X \times Y$, which implies $U \times V$ is open in $X \times Y$. Let $\mathscr{B}_M = \{U \times V|U \in \tau_X, V \in \tau_Y\}$.

Proposition 1.42. *$\mathscr{B}_M$ is a topological basis for $\tau_{X\times Y}$.*

Proof. $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. □

Definition 1.43. The topology generated τ by $\mathscr{B}_M$ is called the **product topology** and the topological space $(X \times Y, \tau)$ is called the **product space** of X and Y, denoted by $X \times Y$.

Given topological spaces $X_i, 1 \leqslant i \leqslant n$, the product space $X_1 \times X_2 \times \cdots \times X_n$ can be defined similarly. $\widetilde{X} = \prod_{i\in\Lambda} X_i = \{(x_i)_{i\in I}|x_i \in X_i\}$, where I is an index set. Let $p_j : \widetilde{X} \to X_j$ be the projection defined by $p_j((x_i)_{i\in I}) = x_j$. Let $\mathscr{B}_P = \{\bigcap_{i=1}^n p_{j_i}^{-1}(U_{j_i})|n \in \mathbb{N}, j_i \in I, U_{j_i} \in \tau_{X_{j_i}}\}$.

Definition 1.44. For infinite topological spaces, we have two kinds of product topology. The topology generated by $\mathscr{B}_P$ is called the **product topology** and the topological space $(\widetilde{X}, \tau)$ is called the **product space** of $\{X_i\}_{i\in I}$, denoted by $\prod_{i\in I} X_i$. The topology generated by $\mathscr{B}_M = \{\prod_{i\in I} U_i|U_i \in \tau_{X_i}, U_i = X_i$ for all but finitely many $i\}$ is called the **box topology**.

The product topology is the smallest topology such that all the projections are continuous.

Proposition 1.45. *Let X, Y, Z be topological spaces and $f : Z \to X \times Y$ be a map. Then f is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous.*

Proof. $\Rightarrow$: By Proposition 1.32, both $p_X \circ f$ and $p_Y \circ f$ are continuous.

$\Leftarrow$: For every open set $W \subseteq X \times Y$, and $z \in f^{-1}(W)$, there exists $U \in \tau_X$ and $V \in \tau_Y$ such that $f(z) \in U \times V \subseteq W$. Then $z \in (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(V) \subseteq f^{-1}(W)$. By Proposition 1.4, $f^{-1}(W)$ is open in Z. $f^{-1}(U \times V) = f^{-1}((U \times Y) \cap (X \times V)) = (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(Y)$ is open in Z. $z \in f^{-1}(U \times V) \subseteq f^{-1}(W)$ and z is an interior point of $f^{-1}(W)$. Therefore, $f^{-1}(W)$ is open in Z. □

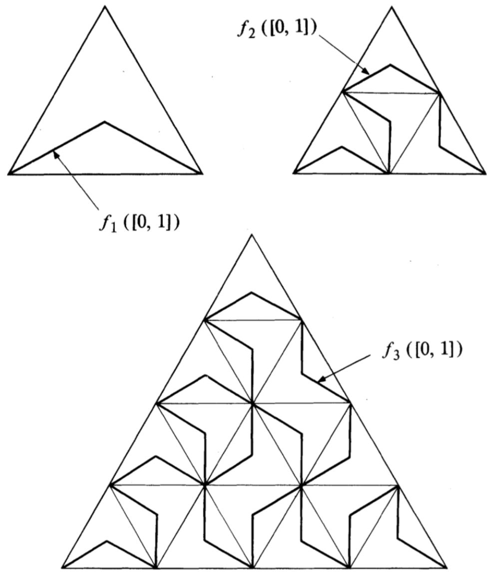
1.5 Space-filling Curves

Now we introduce Peano curves, Hilbert curves and Sierpinski triangles.

Let Δ be an equilateral triangle and we construct a sequence of continuous functions $f_n : [0, 1] \to \Delta$ as shown.

Suppose $n \geqslant m$, then given $t \in [0, 1]$, we can find a small triangle which contains both $f_n(t)$ and $f_m(t)$ and whose sides have length $\frac{1}{2^m}$. Hence $\|f_n(t), f_m(t)\| \leqslant \frac{1}{2^m}$, which proves $\{f_n(t)\}$ is uniformly convergent to f. Since each f_n is continuous, f is also continuous.

Given a point x of Δ together with a neighborhood U of $x \in \mathbb{E}^2$. Choose N large enough so that $B(x, \frac{1}{2^N}) \subseteq U$. And choose $t_0 \in [0, 1]$


f_2 ([0, 1])
f_1 ([0, 1])
f_3 ([0, 1])

2 Several Topological Properties

2.1 Separation Axioms and Countability Axioms

Definition 2.1. A topological space X is a $\boldsymbol{T_1}$ **space** if for every $x, y \in X$ and $x \neq y$, there exists a neighborhood U of x such that $y \notin U$ and a neighborhood V of y such that $x \notin V$.

X is a $\boldsymbol{T_2}$ or **Hausdorff space** if for every $x, y \in X$ and $x \neq y$, there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

X is a $\boldsymbol{T_3}$ or **regular space** if for every x and closed set A such that $x \notin A$, they have disjoint neighborhoods.

X is a $\boldsymbol{T_4}$ or **normal space** if for every two disjoint closed sets A and B, they have disjoint neighborhoods.

Apparently, $T_2 \Rightarrow T_1$, but $T_1 \nRightarrow T_2$. $(\mathbb{R}, \tau_f)$ is an example.

Proposition 2.2. *X is a T_1 space $\Longleftrightarrow \forall x \in X$, $\{x\}$ is closed.*

Proof. $\Rightarrow$: $\forall y \in X$ and $y \neq x$, there exists an open neighborhood U_y of y such that $x \notin U_y$. Let $U = \bigcup_{y \in X \backslash \{x\}} U_y$. Then U is open and $U = X \backslash \{x\}$. Hence $\{x\}$ is closed.

$\Leftarrow$: $\forall x, y \in X$ and $x \neq y$, $x \in X \backslash \{y\}$ is open and $y \in X \backslash \{x\}$ is open. □

Corollary 2.3. *Let X be a T_1 space and $A \subseteq X$. If x is a limit point of A, then every neighborhood U of x contains infinitely many points of A.*

Proof. Suppose that there exists an open neighborhood U of x such that U is finite. Then $B = (U \cap A) \backslash \{x\}$ is finite, hence closed by Proposition 2.2. $U \backslash B = U \cap B^c$ is also an open neighborhood of x, but it contains no point of $A \backslash \{x\}$, which contradicts the definition of limit points. □

Proposition 2.4. *A sequence does not converge to two different points in a Hausdorff space.*

Proof. Suppose that $\{x_n\}$ is a sequence and $x_n \to x$. For every $x_0 \neq x$, there exist disjoint open sets U and V such that $x_0 \in U$ and $x \in V$. Then $\{n \in \mathbb{N} | x_n \notin U\}$ is finite, hence $\{n \in \mathbb{N} | x_n \in V\}$ is finite. By Corollary 2.3, $x_n \nrightarrow x_0$. □

Hausdorff condition helps avoid some weird examples. For example, consider $X = (\mathbb{R} \backslash \{0\}) \cup \{z_1, z_2\}$. Define $p_i : \mathbb{E}^1 \to X$ by $p_i(x) = \begin{cases} x, & x \neq 0 \\ z_i, & x = 0 \end{cases}$, $i = 1, 2$. Give the largest topology such that both p_1 and p_2 are continuous. Then z_1 and z_2 have no disjoint neighborhood, hence X is not Hausdorff. But X is T_1 space.

Proposition 2.5. *X is a T_3 space $\Longleftrightarrow \forall x \in X$ and every open neighborhood W of x, there exists a neighborhood U of x such that $\overline{U} \subseteq W$.*

X is a T_4 space $\Longleftrightarrow \forall$ closed set A and every open neighborhood W of A, there exists a neighborhood U of A such that $\overline{U} \subseteq W$.

Proof. For the sufficiency, if A is a closed set or singleton and B is closed and disjoint with A, then B^c is an open neighborhood of A. There exists a neighborhood U of A such that $A \subseteq \overline{U} \subseteq B^c$. Let $V = \overline{U}^c$, then $B \subseteq V$. Hence U and V are disjoint neighborhoods of A and B.

For the necessity, if X is a T_3 or T_4 space, then A and B have disjoint neighborhoods U and V. By V^c is closed and $U \subseteq V^c$, we have $\overline{U} \subseteq V^c$. And since $W^c \subseteq V$, we have $V^c \subseteq W$. This shows that $A \subseteq \overline{U} \subseteq W$. □

We can notice that $T_1 + T_4 \Rightarrow T_3$ and $T_1 + T_3 \Rightarrow T_2$, where T_1 is necessary.

Example 2.6. Given $\tau = \{(-\infty, a) | -\infty \leqslant a \leqslant +\infty\}$, $(\mathbb{R}, \tau)$ is normal, but closed sets $[a, \infty)$ always intersect. Then $(\mathbb{R}, \tau)$ is not T_1, T_2 or T_3.

Consider $A \subseteq X$ is a subspace topology, then X is T_1, T_2 or $T_3 \Rightarrow A$ is T_1, T_2 or T_3. But T_4 is not hereditary.

Example 2.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Every nonempty closed set contains d, hence X is T_4.

Let $Y = \{a, b, c\}$ and $\tau_Y = \{\varnothing, Y, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. $\{a\}$ and $\{c\}$ are two disjoint closed sets in Y, but they have no disjoint neighborhoods. Hence Y is not T_4.

Proposition 2.8. *$X \times Y$ is Hausdorff $\Longleftrightarrow$ both X and Y are Hausdorff.*

Proof. $\Rightarrow$: $\forall x_1, x_2 \in X$ and $x_1 \neq x_2$, let $y_0 \in Y$. Then $(x_1, y_0) \neq (x_2, y_0)$. There exist disjoint open sets W_1 and W_2 in $X \times Y$ and there exists open sets $U_1, U_2 \in X$ and $V_1, V_2 \in Y$ such that $(x_i, y_0) \in U_i \times V_i \subseteq W_i, i = 1, 2$. Since $(U_1 \times V_1) \cap (U_2 \times V_2) = \varnothing$ and $y_0 \in V_1 \cap V_2$, U_1 and U_2 are disjoint open sets in X. Similarly, Y is Hausdorff.

$\Leftarrow$: For every $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$. WLOG, assume that $x_1 \neq x_2$. x_1 and x_2 have disjoint open neighborhoods U_1 and U_2. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint open neighborhoods of (x_1, y_1) and (x_2, y_2). □

Definition 2.9. (X, τ) is **first countable** (a C_1 space) if every point has a countable neighborhood basis, i.e. $\forall x \in X$, there exists countable open neighborhoods $U_i, i \in \mathbb{N}$ such that for every open neighborhood V of x, there exists i such that $x \in U_i \subseteq V$.

Example 2.10. $\mathbb{E}^k$ is first countable. Since for every point $x \in \mathbb{E}^k$, let $U_n = B(x, \frac{1}{n}), n \in \mathbb{N}$, then $\{U_n | n \in \mathbb{N}\}$ is a countable neighborhood basis of x.

Example 2.11. $(\mathbb{R}, \tau_f)$ is not first countable. For any countable collection of open neighborhoods $\{U_n | n \in \mathbb{N}\}$ of 0, U_n^c are finite. Then $\left(\bigcap_{n=1}^{\infty} U_n\right)^c = \bigcup_{n=1}^{\infty} U_n^c$ is at most countable. Then $\bigcap_{n=1}^{\infty} U_n$ is uncountable. For every $x_0 \in \bigcap_{n=1}^{\infty} U_n \backslash \{0\}$, $\mathbb{R} \backslash \{x_0\}$ is an open neighborhood of 0, but for every n, $U_n \nsubseteq \mathbb{R} \backslash \{x_0\}$. Hence $(\mathbb{R}, \tau_f)$ is not first countable.

Definition 2.12. (X, τ) is **second countable** (a C_2 space) if X has a countable topological basis.

Example 2.13. $\mathbb{E}^n$ is C_2. Choose the set of all open balls with rational center and rational radius as the topological basis.

Proposition 2.14. *$C_2 \Longrightarrow C_1$+separable.*

Proof. X is C_2 and $\mathscr{B}$ is a countable topological basis. $\forall x \in X$, let $\mathscr{B}' = \{U \in \mathscr{B} | x \in U\}$, then $\mathscr{B}'$ is a countable collection of neighborhoods of x. For every neighborhood V of x, there exists $U \in \mathscr{B}$ such that $x \in U \subseteq V$. Hence $\mathscr{B}'$ is a countable neighborhood basis of x. Therefore, X is C_1. □

Example 2.15. In $\mathbb{R}$, define $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$. For every $x \in \mathbb{R}$, $\{\{x\}\}$ is a countable neighborhood basis of x. Hence $(\mathbb{R}, d)$ is C_1, but not C_2.

Define $d_p(x,y) = \begin{cases} 0, & x = y \\ |x| + |y|, & x \neq y \end{cases}$. $(\mathbb{R}, d_p)$ is C_1, but not C_2.

Theorem 2.16 (Lindelöf Theorem)**.** $T_3 + C_2 \Longrightarrow T_4$.

Proof. Let X be a T_3 and C_2 space, then X has a countable neighborhood basis $\mathscr{B}$. E and E' be two disjoint closed sets in X. Assume $\{B_1, B_2, \cdots\}$ are all open sets in $\mathscr{B}$ such that $\overline{B_i} \cap E = \varnothing$, and $\{B'_1, B'_2, \cdots\}$ are all open sets in $\mathscr{B}$ such that $\overline{B'_i} \cap E' = \varnothing$.

Let $U_n = B_n \backslash \bigcup_{i=1}^{n-1} \overline{B'_i}$ and $V_n = B'_n \backslash \bigcup_{i=1}^{n-1} \overline{B_i}$. Then U_n and V_m are open and disjoint. Let $U = \bigcup_{n=1}^{\infty} U_n$ and $V = \bigcup_{n=1}^{\infty} V_n$, then $U \cap V = \varnothing$. $\forall x \in E$, x is disjoint with E', hence there exists an open neighborhood W of x such that $\overline{W} \cap E' = \varnothing$. There exists $B_n \in \mathscr{B}$ such that $x \in B_n \subseteq W$. Then $\overline{B_n} \cap E' = \varnothing$, hence B_n is in the list. Then $x \in U_n \subseteq U$. Similarly, $E' \subseteq V$. Therefore, X is normal. □

Definition 2.17. $\{x_n\}$ in (X, d) is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall m, n > N$, $d(x_n, x_m) < \varepsilon$.

(X, d) is **complete** if every Cauchy sequence converges to a point in X.

Proposition 2.18. *A metric space satisfies T_i axioms, $i = 1, 2, 3, 4$.*

Proof. E, F are closed sets or singletons and $E \cap F = \varnothing$. $\forall x \in E$, denote $d_E = \inf_{y \in E} d(x,y)$. If E is a singleton, then $d(x, E) = 0 \Leftrightarrow x \in E$. If E is closed and $x \notin E$, then E^c is an open neighborhood of x. There exists $\delta > 0$ such that $B(x, \delta) \subseteq E^c$. $\forall y \in E$, we have $d(x,y) > \frac{\delta}{2}$. Then $d(x, E) \geqslant \frac{\delta}{2} > 0$.

Since $E \cap F = \varnothing$, we have $d(x, E) + d(x, F) > 0$. Define a continuous function $g : X \to \mathbb{E}^1$ by $g(x) = \frac{d(x,E)}{d(x,E)+d(x,F)}$. Then $g(x) = 0 \Leftrightarrow x \in E$ and $g(x) = 1 \Leftrightarrow x \in F$. For every $x \in X \backslash (E \cup F)$, $0 < g(x) < 1$. Let $U = g^{-1}((-\infty, \frac{1}{4}))$ and $V = g^{-1}((\frac{3}{4}, +\infty))$. Then U and V are disjoint open neighborhoods of E and F. □

Proposition 2.19. *A separable metric space X is C_2.*

Proof. Assume A is a countable dense subset of X. Let $\mathscr{B} = \{B(a, \frac{1}{n}) | a \in A, n \in \mathbb{N}^*\}$. Then $\mathscr{B}$ is countable. For every open set $U \subseteq X$ and $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$, then there exists $a \in A$ such that $d(x, a) < \frac{1}{N}$. Then $x \in B(a, \frac{1}{N}) \in \mathscr{B}$. $\forall y \in B(a, \frac{1}{N})$, $d(x,y) \leqslant d(x,a) + d(a,y) < \frac{2}{N} < \varepsilon$. Hence $y \in B(x, \varepsilon) \subseteq U$. Therefore, $B(a, \frac{1}{N}) \subseteq U$. This shows that $U = \bigcup_{x \in U} B_x$ and $\mathscr{B}$ is a countable topological basis of X. □

Example 2.20. $E^\omega = \{\{x_n\} | x_n \in \mathbb{R}, \sum_{n=1}^{\infty} x_n^2 < \infty\}$ and $< \{x_n\}, \{y_n\} >= \sum_{n=1}^{\infty} x_n y_n$. $d(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$. Then E^ω is C_2.

Let $A = \{\{x_n\} | x_n \in \mathbb{Q}, \sum_{n=1}^{\infty} x_n^2 < \infty$ and $\exists N$ such that $n > N, x_n = 0\}$. Then $E^n \subseteq E^\omega$ and A is dense.

2.2 Urysohn's Lemma

Theorem 2.21 (Urysohn's Lemma)**.** *Let X be a normal space and A, B be two disjoint closed sets in X. There exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Let $\mathbb{Q}_0 = [0,1] \cap \mathbb{Q}$ and $Q_0 = \{r_1 = 1, r_2 = 0, r_3, \cdots\}$. First, for every $r_i \in \mathbb{Q}_0$, construct an open set U_i such that

(i) $r_n < r_m \Longrightarrow U_n \subseteq U_m$;

(ii) $A \subseteq U_n \subseteq B^c$.

Let $U_1 = B^c$. By the normality of X, A and B have disjoint open neighborhoods U_2 and V. Then $\overline{U_2} \subseteq B^c$. Suppose that $U_1, U_2, \cdots, U_k$ have been constructed. $r_m = \max\{r_l | l \leqslant n, r_l < r_{n+1}\}$ and $r_k = \min\{r_l | l \leqslant n, r_l > r_{n+1}\}$. By the normality of X, $\overline{U_m}$ and U_k^c have disjoint open neighborhoods U_{n+1} and V_{n+1}. Then $\overline{U_m} \subseteq U_{n+1} \subseteq \overline{U_{n+1}} \subseteq V_{n+1}^c$.

Define $f : X \to \mathbb{E}^1$ by $f(x) = \sup\{0, r_n \in Q_0 | x \notin U_n\} = \inf\{1, r_n \in Q_n | x \in U_n\}$. f is continuous at x iff $\forall (a,b) \in \mathbb{E}^1$, if $x \in f^{-1}(a,b)$, then there exists open neighborhood $U_x \subseteq f^{-1}(a,b)$ of x.

If $0 < f(x) < 1$, then there exists $r_m, r_k \in Q_0$ such that $a < r_m < f(x) < r_k < b$. Then $x \in U_k \backslash \overline{U_m} \subseteq f^{-1}(r_m, r_k) \subseteq f^{-1}(a,b)$. If $f(x) = 0$, then for every $r_n \in Q_0$ and $n \geqslant 2$, $x \in U_n$. Let $U_x = U_2$, then $U_x \subseteq f^{-1}(0, r_2) \subseteq f^{-1}(a,b)$. If $f(x) = 1$, then there exists $r_n \in Q_0$ such that $x \notin U_n$. Let $U_x = U_n^c$, then $U_x \subseteq f^{-1}(r_n, 1) \subseteq f^{-1}(a,b)$.

If $f(x) = 0$, choose $r_k \in \mathbb{Q}_0$ such that $0 < r_k < b$. Then $x \in U_k \subseteq f^{-1}(a,b)$.

If $f(x) = 1$, choose $r_m \in \mathbb{Q}_0$ such that $a < r_m < 1$. Then $x \in \overline{U_m}^c \subseteq f^{-1}(a,b)$. □

If x is a metric space, $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ is a continuous function that satisfies the conditions of Urysohn's Lemma.

Theorem 2.22 (Tietze Extension Theorem)**.** *Let X be a normal space and $E \subseteq X$ be closed. If $f : E \to \mathbb{E}^1$ is continuous, then there exists a continuous function $\overline{f} : X \to \mathbb{E}^1$ such that $\overline{f}|_E = f$.*

Proof. If f is bounded, WLOG, assume $f(E) \subseteq [0,1]$. Let $A_1 = f^{-1}([-1, -\frac{1}{3}])$ and $B_1 = f^{-1}([\frac{1}{3}, 1])$, then A_1 and B_1 are disjoint closed sets. By Urysohn's Lemma, there exists a continuous function $\varphi_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $\varphi_1(A_1) = \{-\frac{1}{3}\}$ and $\varphi_1(B_1) = \{\frac{1}{3}\}$. Let $f_1 = f - \varphi_1$, then $f_1(E) \subseteq [-\frac{2}{3}, \frac{2}{3}]$.

Define $\varphi_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ and $f_2 = f_1 - \varphi_2$. We have $f_2(E) \subseteq [-\frac{4}{9}, \frac{4}{9}]$. By induction, we can define $\{\varphi_n : X \to \mathbb{E}^1\}$ such that $\varphi_n(x) = [-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}]$ and $|f(x) - \varphi_1(x) - \cdots - \varphi_n(x)| \leqslant \left(\frac{2}{3}\right)^n$ for every $x \in E$. By M-test, $\sum_{n=1}^{\infty} \varphi_n$ converges uniformly to a continuous function $\widetilde{f}$, which satisfies $\widetilde{f}|_E = f$ and $\widetilde{f}(X) \subseteq [-1, 1]$.

For general continuous function $f : E \to \mathbb{E}^1$. Denote $g = \arctan(f(x))$, then $g(E) \subseteq (-1, 1)$. By the above argument, there exists a continuous function $\widetilde{g} : X \to [-1, 1]$. Let $F = (g)^{-1}(\{\pm 1\})$ be a closed set. Then $E \cap F = \varnothing$. By Urysohn's Lemma, there exists a continuous function $h : X \to [0,1]$ such that $h(E) = \{0\}$, $h(F) = \{1\}$ and $h(x)\widetilde{g}(x) \in (-1, 1)$. On F, $h(x)\widetilde{g}(x) = 0$. Let $\widetilde{f}(x) = \tan(h(x)\widetilde{g}(x))$, then $\overline{f} : X \to \mathbb{E}^1$ is continuous and $\overline{f}|_E = f$. □

In fact, Tietze Extension Theorem is equivalent to T_4 axiom.

Definition 2.23. (X, τ) is **metrizable** if there exists a metric d on X such that $\tau_d = \tau$.

Example 2.24. $(\mathbb{R}, \tau_c)$ is not metrizable. Let $U_n = \{y \in \mathbb{R} | d(y, 0) < \frac{1}{n}\}$, then U_n is open and $\mathbb{R} \backslash U_n$ is countable. Let $U = \bigcap_{n=1}^{\infty} U_n$, then U^c is countable and U is uncountable. Hence there exist $x, z \in U$ such that $x \neq z$ and $d(x,0) = d(z,0) = 0 \Rightarrow d(x,z) = 0$. This contradicts that $x \neq z$.

Theorem 2.25 (Urysohn Metrization Theorem)**.** *$C_2, T_2, T_4 \Longrightarrow$ metrizable.*

Proof. Let (X, τ) be a C_2, T_4, Hausdorff space and $\mathscr{B}$ be a topological basis of X. Let $P = \{(B, B') \in \mathscr{B} \times \mathscr{B} | \overline{B} \subseteq B'\} \neq \varnothing$, then P is countable. Denote $P = \{(B_1, B_1'), \cdots, (B_n, B_n'), \cdots\}$.

For every n, by Urysohn's Lemma and normality of X, there exists a continuous function $f : X \to [0, 1]$ such that $f(\overline{B_n}) = \{0\}$ and $f(X \backslash B_n') = \{1\}$. For every $x, y \in X$, define

$$d(x, y) = \left(\sum_{n=1}^{\infty} \frac{(f_n(x) - f_n(y))^2}{n^2} \right)^{\frac{1}{2}}.$$

We claim that d is a metric on X. First, $0 \leqslant d(x, y) < \infty$. Assume $x \neq y$, by Hausdorff condition, there exist $B' \in \mathscr{B}$ such that $x \in B'$ but $y \notin B'$. By the normality of X, $\{x\}$ and $X \backslash B'$ have disjoint open neighborhoods U, V.

There exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$ and $\overline{B} \subseteq X \backslash V \subseteq B'$. Then we have $(B, B') \in P$ and there exists n such that $(B, B') = (B_n, B_n')$. Then $f_n(x) = 0$ and $f_n(y) = 1$, hence $d(x, y) \geqslant \frac{1}{n} > 0$. The symmetry and triangle inequality of d are obvious.

Let U be an open set of τ_d. For every $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Choose N such that

$$\sum_{n=N}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{2}.$$

For $i = 1, \cdots, N$, $g_i(y) = |f_i(y) - f_i(x)|$ is continuous. Let $W_i = g_i^{-1}([0, \frac{\varepsilon}{\sqrt{2N}}))$, then $W_i \subseteq (X, \tau)$ is open. Let $W_x = W_1 \cap \cdots \cap W_N$, then $W_x \subseteq (X, \tau)$ is open.

$$\forall y \in W_x, \quad d(x, y) < \sqrt{\frac{\varepsilon^2}{2N} N + \frac{\varepsilon^2}{2}} = \varepsilon.$$

Hence $W_x \subseteq U$ and x is an interior point of U in τ. Therefore, U is open in τ.

Conversely, let $U \subseteq (X, \tau)$ be open. For every $x \in U$, there exists $B' \in \mathscr{B}$ such that $x \in B' \subseteq U$. $\{x\}$ and $X \backslash B'$ have disjoint open neighborhoods N_1 and N_2. There exists $B \in \mathscr{B}$ such that $x \in B \subseteq N_1$ and $\overline{B} \subseteq B'$. There exists $k \in \mathbb{N}$ such that $(B, B') = (B_k, B_k')$. For every $y \in X \backslash B'$, we have $f_k(x) = 0$ and $f_k(y) = 1$, hence $d(x, y) \geqslant \frac{1}{k}$ and $B(x, \frac{1}{k}) \subseteq U$. Therefore, x is an interior point of U in τ_d and U is open in τ_d. ☐

2.3 Compactness

Theorem 2.26 (Heine-Borel Theorem)**.** *Any open cover of $[0, 1]$ has a finite subcover.*

Proof. Let $\mathscr{C}$ be an open cover of $[0, 1]$. Define $F = \{a \in [0, 1] | [0, a]$ can be covered by finite open sets in $\mathscr{C}\}$.

- $0 \in F$, since there exists $U \in \mathscr{C}$ such that $0 \in U$.
- $a \in \mathscr{C} \Longrightarrow$ for every $b \in [0, a]$, we have $b \in F \Longrightarrow F$ is an interval

Assume $F = [0, A]$, then there exists $U \in \mathscr{C}$ such that $A \in U$. Since U is open, there exists $\varepsilon > 0$ such that $(A - \varepsilon, A + \varepsilon) \subseteq U$. Then $[0, A - \frac{\varepsilon}{2}]$ can be covered by $U_1, \cdots, U_n$.

If $A < 1$, WLOG suppose $A + \frac{\varepsilon}{2} < 1$, then $[0, A + \frac{\varepsilon}{2}]$ can be covered by $U_1, \cdots, U_n$ and U, which contradicts the definition of A. Therefore, $A = 1$ and $[0, 1]$ can be covered by finite open sets in $\mathscr{C}$. ☐

Proof. Suppose $[0,1]$ can not be covered by finite open sets in $\mathscr{C}$. Then $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ can not be covered by finite open sets in $\mathscr{C}$ and we denote $a_0 = 0$, $b_0 = 1$. If $[0,\frac{1}{2}]$ can not be covered by finite open sets in $\mathscr{C}$, let $a_1 = 0$, $b_1 = \frac{1}{2}$. Otherwise, let $a_1 = \frac{1}{2}$, $b_1 = 1$. Inductively, we can define a sequence of closed intervals $[a_n, b_n]$ such that $b_n - a_n = \frac{1}{2^n}$ and $[a_n, b_n]$ can not be covered by finite open sets in $\mathscr{C}$.

By the Nested Interval Theorem, there exists a unique point $x_* \in [a_n, b_n]$. There exists $U \in \mathscr{C}$ such that $x_* \in U$ and there exists $\varepsilon > 0$ such that $(x_* - \varepsilon, x_* + \varepsilon) \subseteq U$. Choose N such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$, then $[a_N, b_N] \subseteq (x_* - \varepsilon, x_* + \varepsilon) \subseteq U$, which contradicts the construction of $[a_N, b_N]$. □

Definition 2.27. X is **compact** if every open cover of X has a finite subcover.

Heine-Borel Theorem shows that $[0,1]$ is compact.

Definition 2.28. $E \subseteq X$ is a **compact subset** if every open cover of E has a finite subcover.

- If X or its topology is finite, then X is compact.
- $(\mathbb{R}, \tau_f)$ is compact and $\mathbb{E}^1$ is not compact.
- If $X \cong Y$, then X is compact $\Longleftrightarrow$ Y is compact.

Proposition 2.29. *The image of a compact set under a continuous map is compact.*

Proof. Let $E \subseteq X$ be compact and $f : X \to Y$ be continuous. For every open cover $\mathscr{C}$ of $f(E)$, $\mathscr{C}' = \{f^{-1}(U) | U \in \mathscr{C}\}$ is an open cover of E. There exists a finite subcover $\{f^{-1}(U_1), \cdots, f^{-1}(U_n)\}$, then $\{U_1, \cdots, U_n\}$ is a finite subcover of $f(E)$. Therefore, $f(E)$ is compact. □

Proposition 2.30. *A closed subset of a compact space is compact.*

Proof. Let $E \subseteq X$ be closed and X be compact. For every open cover $\mathscr{C}$ of E, $\mathscr{C}' = \mathscr{C} \cup \{E^c\}$ is an open cover of X. There exists a finite subcover $\{U_1, \cdots, U_n, E^c\}$, then $\{U_1, \cdots, U_n\}$ is a finite subcover of E. Therefore, E is compact. □

Example 2.31. Cantor set is compact.

Proposition 2.32. *Continuous real-valued function on a compact space is bounded.*

Proof. Let $f : X \to \mathbb{E}^1$ be continuous and X be compact, then $f(X)$ is compact. Take the open cover $\{(-n, n) | n \in \mathbb{N}^*\}$, then there exists a finite subcover $\{(-n_1, n_1), \cdots, (-n_k, n_k)\}$. Let $N = \max\{n_1, \cdots, n_k\}$, then $f(X) \subseteq (-N, N)$. □

Proposition 2.33. $E \subset \mathbb{E}^1$ *is compact* $\Longleftrightarrow$ E *is closed and bounded.*

Proof. $\Longleftarrow$: Let $E \subset [-N, N]$. By Heine-Borel Theorem, $[-N, N]$ is compact. Since E is closed subset of $[-N, N]$, E is compact.

$\Longrightarrow$: Let $f : E \to \mathbb{E}^1$ be the inclusion map. Since f is continuous, by Proposition 2.32, E is bounded. Now we show that E is closed.

Suppose E is not closed and $x_0 \notin E$ is a limit point of E. Then $\{U_n = (-\infty, x_0 - \frac{1}{n}) \cup (x_0 + \frac{1}{n}, +\infty)\}$ is an open cover of E and $U_{n_1}, \cdots, U_{n_k}$ is a finite subcover. Denote $N = \max\{n_1, \cdots, n_k\}$, then $E \subseteq U_N$ and $(x_0 - \frac{1}{N}, x_0 + \frac{1}{N}) \cap E = \varnothing$, which contradicts that x_0 is a limit point of E. □

In general, compact sets can be not closed and the closure of compact sets can be not compact.

Example 2.34.
- X has more than two element and $\tau = \{\varnothing, X\}$. Then the singleton subsets of X are not closed, but they are compact.
- For $(\mathbb{R}, \tau_f)$, $\mathbb{Z}$ is not closed, but is compact.

Proposition 2.35. *If X is Hausdorff and $A \subseteq X$ is compact, then for every $y \in A^c$, A and $\{y\}$ have disjoint open neighborhoods. Hence, A is closed.*

Proof. For every $x \in A$, there exist disjoint open neighborhoods U_x of x and V_x of y. Then $\{U_x | x \in A\}$ is an open cover of A and there exists a finite subcover $\{U_{x_1}, \cdots, U_{x_n}\}$. Let $U = U_{x_1} \cup \cdots \cup U_{x_n}$ and $V = V_{x_1} \cap \cdots \cap V_{x_n}$, then U and V are disjoint open neighborhoods of A and $\{y\}$ respectively.

For every $y \in A^c$, there exists an open neighborhood V_y of y such that $y \in V_y \subseteq A^c$. Hence, A^c is open and A is closed. □

Proposition 2.36. *The continuous bijection f from a compact space X to a Hausdorff space Y is a homeomorphism.*

Proof. It suffices to show that f^{-1} is continuous, i.e. for every closed set $E \subseteq X$, $(f^{-1})^{-1}(E) = f(E)$ is closed. Since X is compact, E and $f(E)$ is compact. By Proposition 2.35, $f(E)$ is closed. □

Proposition 2.37. *Two disjoint compact subsets A, B of a Hausdorff space X have disjoint open neighborhoods.*

Corollary 2.38. *Compact Hausdorff space is T_3 and T_4.*

Proposition 2.39 (Bolzano-Weierstrass Theorem)**.** *Infinite subsets of compact space X have a limit point.*

Proof. Suppose A is an infinite subset of X and has no limit point. For every $x \in X$, there exists an open neighborhood U_x of x such that $U_x \cap A$ is finite. Then $\{U_x | x \in X\}$ is an open cover of X. There exists a finite subcover $\{U_{x_1}, \cdots, U_{x_n}\}$, then A is finite, which contradicts that A is infinite. □

Definition 2.40. X is **limit point compact** if every infinite subset of X has a limit point.

Definition 2.41. X is **sequentially compact** if every sequence in X has a convergent subsequence.

Example 2.42. $(\mathbb{R}, \tau_f)$ is sequentially compact.

Proposition 2.43. *Continuous real-valued function on a sequentially compact space X is bounded and attains its maximum and minimum.*

Proof. If f is not bounded, then there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to x_*.

Let $g(x) = |f(x)|$, then $U = g^{-1}(g(x_*) - 1, g(x_*) + 1)$ is an open neighborhood of x_*. There exists N such that for every $k > N$, $x_{n_k} \in U$, i.e. $g(x_*) - 1 < g(x_{n_k}) < g(x_*) + 1$, which contradicts that $\lim_{k\to\infty} |f(x_{n_k})| = \infty$. □

Proposition 2.44. *Compact C_1 space X is sequentially compact.*

Proof. For infinite sequence $\{x_n\}$ of X, by Bolzano-Weierstrass Theorem, there exists $x_0 \in X$ such that every open neighborhood of x_0 contains infinitely many elements of $\{x_n\}$. Take the countable neighborhood basis $\{U_n\}$ of x_0 such that for $m > n$, $U_m \subseteq U_n$. Choose n_1 such that $x_{n_1} \in U_1$ and choose $n_2 > n_1$ such that $x_{n_2} \in U_2$ and so on. Then $\{x_{n_k}\}$ converges to x_0. □

Theorem 2.45. *If X is a metric space, then X is compact $\Longleftrightarrow$ X is sequentially compact.*

Proof. $\Rightarrow$: Let $\{x_n\}$ be an infinite sequence of X and WLOG, assume $x_m \neq x_n$ for $m \neq n$. By Bolzano-Weierstrass Theorem, there exists a limit point x_0 of $\{x_n\}$. For every $k \in \mathbb{N}$, $B(x_0, \frac{1}{k}) \cap \{x_n\}$ is infinite. Choose $x_{n_1} \in B(x_0, 1) \cap \{x_n\}$ and if $x_{n_1}, \cdots, x_{n_{k-1}}$ have been chosen, choose $x_{n_k} \in B(x_0, \frac{1}{k}) \cap \{x_n\}$ such that $n_k > n_{k-1}$. Then $\{x_{n_k}\}$ converges to x_0.

$\Leftarrow$: Let $\mathscr{C}$ be an open cover of X. We claim that there exists $\delta > 0$, which is called the **Lebesgue number**, such that for every $x \in X$, $\exists U \in \mathscr{C}$ such that $B(x, \delta) \subseteq U$.

Otherwise, for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that for every $U \in \mathscr{C}$, $B(x_n, \frac{1}{n}) \backslash U \neq \varnothing$. Then $\{x_n\}$ has a convergent subsequence convergent to x_*, then there exists $U_* \in \mathscr{C}$ such that $x_* \in U_*$. There exists $\delta_* > 0$ such that $B(x_*, \delta_*) \subseteq U_*$. $B(x_*, \delta_*) \cap \{x_n\}$ is infinite and there exists n_0 such that $x_{n_0} \in B(x_*, \frac{\delta_*}{2})$ and $\frac{1}{n_0} < \frac{\delta_*}{2}$. Then for every $y \in B(x_{n_0}, \frac{1}{n_0})$, $d(y, x_*) \leqslant d(y, x_{n_0}) + d(x_{n_0}, x_*) < \frac{1}{n_0} + \frac{\delta_*}{2} < \delta_*$. Hence, $B(x_{n_0}, \frac{1}{n_0}) \subseteq B(x_*, \delta_*) \subseteq U_*$, which contradicts the construction of x_{n_0}.

We also claim that there exists a finite subset $A = \{z_1, \cdots, z_m\} \subseteq X$ such that for every $x \in X$, there exists z_i such that $d(x, z_i) < \delta$.

Otherwise, there exists infinite sequence $\{x_n\}$ such that for every $m \neq n$, $d(x_m, x_n) \geqslant \delta$. Then $\{x_n\}$ has no convergent subsequence, which contradicts that X is sequentially compact.

Hence $\{B(z_i, \delta)\}$ covers X. For every z_i, take $U_i \in \mathscr{C}$ such that $B(z_i, \delta) \subseteq U_i$. Then $\{U_1, \cdots, U_m\}$ is a finite subcover of $\mathscr{C}$. Therefore, X is compact. □

Proposition 2.46 (Lebesgue Lemma)**.** *If X is a compact metric space and $\mathscr{C}$ is an open cover of X, then there exists $\delta > 0$ such that every subset of X with diameter less than δ is contained in some member of $\mathscr{C}$.*

Lemma 2.47. *Let A be a compact subset of X, $y \in Y$ and W is a neighborhood of $A \times \{y\}$. Then there exists neighborhoods U of A and V of y such that $U \times V \subseteq W$.*

Proof. Since A is compact, $A \times \{y_0\}$ is compact. For every $(x, y_0) \in A \times \{y_0\} \subseteq W$, there exists open neighborhoods U_x of x and V_x of y_0 such that $(x, y_0) \in U_x \times V_x \subseteq W$. Hence, $\{U_x \times V_x | x \in A\}$ is an open cover of $A \times \{y_0\}$. There exists a finite subcover $\{U_{x_1}, \cdots, U_{x_n}\}$. Let $U = U_{x_1} \cup \cdots \cup U_{x_n}$ and $V = V_{x_1} \cap \cdots \cap V_{x_n}$, then $A \times \{y_0\} \subseteq U \times V \subseteq W$. Therefore, U and V are open neighborhoods of A and y_0 respectively. □

Proposition 2.48. *$X \times Y$ is compact $\Longleftrightarrow$ both X and Y are compact.*

Proof. $\Rightarrow$: Since $Pr_X : X \times Y \to X$ and $Pr_Y : X \times Y \to Y$ are continuous, X and Y are compact.

$\Leftarrow$: For every $y \in Y$, $X \times \{y\}$ is compact, so there exists finite subsets $U_{y,1}, \cdots, U_{y,n_y}$ of $\mathscr{C}$ which covers $X \times \{y\}$. Let $W_y = U_{y,1} \cup \cdots \cup U_{y,n_y}$ be an open neighborhood of $X \times \{y\}$. By Lemma 2.47, there exists open set $V_y \subseteq Y$ such that $X \times \{y\} \subseteq X \times V_y \subseteq W_y$. Then $\{V_y | y \in Y\}$ is an open cover of Y and there exists a finite subcover $\{V_{y_1}, \cdots, V_{y_m}\}$. Let $\mathscr{C}' = \bigcup_{i=1}^{m} \{U_{y_i,1}, \cdots, U_{y_i,n_{y_i}}\}$, then $\mathscr{C}'$ is a finite subcover of $\mathscr{C}$. Therefore, $X \times Y$ is compact. □

By induction, we have that the finite product of compact spaces is compact. For infinite product, we need the following theorem.

Theorem 2.49 (Tychonoff Theorem)**.** *$X_i, i \in \Lambda$ is a collection of nonempty compact spaces, then $X = \prod_{i \in \Lambda} X_i$ is compact in the product topology.*

Kelly showed that Tychonoff Theorem $\iff$ Axiom of Choice.

Definition 2.50. X is **locally compact** if for every $x \in X$, there exists a compact neighborhood of x.

For example, compact space and $\mathbb{E}^n$ are locally compact.

Proposition 2.51. *X is a locally compact Hausdorff space, then we have*

1. *X is T_3;*
2. *$\forall x \in X$, the compact neighborhoods of x form a neighborhood basis of x;*
3. *Open subsets of X are locally compact.*

Definition 2.52. The open cover $\mathscr{C}$ of X is called **locally finite** if every point of X has a neighborhood V such that $\#\{U \in \mathscr{C} | U \cap V \neq \varnothing\} < \infty$.

$\mathscr{U}$ and $\mathscr{U}'$ are open covers of X. $\mathscr{U}'$ is a **refinement** of $\mathscr{U}$ if $\forall U' \in \mathscr{U}'$, there exists $U \in \mathscr{U}$ such that $U' \subseteq U$.

Definition 2.53. X is **paracompact** if every open cover of X has a locally finite open refinement.

Example 2.54. A compact space is paracompact. $\mathbb{E}^n$ and $[-n, n]^n$ are paracompact.

Proposition 2.55. *A paracompact Hausdorff space is normal.*

Proposition 2.56. *X is a locally compact C_2 Hausdorff space. Then every open cover of X has countable locally finite open refinement.*

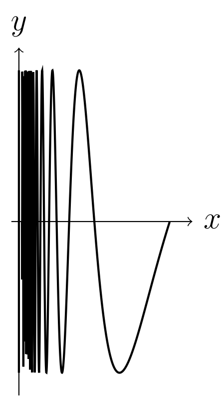
Proposition 2.57. *Metric spaces are paracompact.*

The proof uses the Axiom of Choice.

2.4 Connectedness

Example 2.58. $X = A \cup B$ is connected, where

$$A = \left\{ \left(x, \sin\frac{\pi}{x}\right) \middle| x \in (0, 1] \right\}, \quad B = \{(0, y) | -1 \leqslant y \leqslant 1\}$$



Definition 2.59. X is **connected** if for every division $X = A \cup B$ such that A and B are nonempty, then $A \cap \overline{B} \neq \varnothing$ or $\overline{A} \cap B \neq \varnothing$.

Example 2.60. $[0, 1]$ is connected.

Theorem 2.61. *$E \subseteq \mathbb{E}^1$ is connected $\iff$ E is an interval.*

Proof. $\Rightarrow$: I is an inverval and let $I = A \cup B$ such that A and B are nonempty. Suppose that $A \cap B = \varnothing$. Choose $a \in A$ and $b \in B$, WLOG, assume $a < b$. Let $s = \sup\{x \in A | x < b\}$.

If $s \in A$, then $s < b \Rightarrow (s, b] \subseteq B \Rightarrow s \in \overline{B} \Rightarrow A \cap \overline{B} \neq \varnothing$. If $s \in B$, then $s \in \overline{A} \Rightarrow \overline{A} \cap B \neq \varnothing$. Hence I is connected.

$\Leftarrow$: It suffices to show that for every $a < b, a, b \in E$, every c such that $a < c < b$ is in E. Otherwise, there exist $p \in (a, b)$ such that $p \notin E$. Let $A = E \cap (-\infty, p)$ and $B = E \cap (p, +\infty)$, then $a \in A$ and $b \in B$. $\overline{A} \cap B \subseteq \overline{(-\infty, p)} \cap (p, +\infty) = \varnothing$ and $A \cap \overline{B} \subseteq (-\infty, p) \cap \overline{(p, +\infty)} = \varnothing$. This contradicts the connectedness of E. □

We give another definition of connectedness.

Definition 2.62. X is **connected** if X cannot be represented as the union of two nonempty disjoint open sets.

Proposition 2.63. *X is a topological space. The following statements are equivalent.*

1. *X is connected;*
2. *X and $\varnothing$ are the only subsets of X which are both open and closed;*
3. *X cannot be represented as the union of two nonempty disjoint closed sets.*
4. *The image of a continuous function from X to a discrete space is a singleton.*

Proof. (1) $\Rightarrow$ (2): $A \subseteq X$ is open and closed and $A \neq X$, $A \neq \varnothing$. Let $B = X \backslash A$, then $A, B \neq \varnothing$ and $\overline{A} = A, \overline{B} = B$. Then $X = A \cup B$ such that $A \cap \overline{B} = \varnothing = \overline{A} \cap B$. This contradicts the connectedness of X.

(2) $\Rightarrow$ (3): $X = A \cup B$. Suppose that A and B are nonempty and open, and $A \cap B = \varnothing$. Then A and B are closed. This contradicts (2).

(3) $\Rightarrow$ (4): Let Y be a discrete space and $f : X \to Y$ be continuous. Suppose that $f(x)$ contains at least two points y_1 and y_2. Let $A = f^{-1}(\{y_1\})$ and $B = f^{-1}(\{y_2\})$, then A and B are nonempty closed sets and $X = A \cup B$. This contradicts (3).

(4) $\Rightarrow$ (1): If X is not connected, $X = A \cup B$ such that $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. Let $Y = \{y_1, y_2\}$ be a discrete space and define $f : X \to Y$ such that $f(A) = y_1$ and $f(B) = y_2$. Then f is continuous and the image of f contains two points, which contradicts (4). □

Proposition 2.64. *The continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ be continuous surjective, and X be connected. If Y is not connected, then there exist nonempty open sets $U, V \subseteq Y$ such that $Y = U \cup V$ and $U \cap V = \varnothing$.

Let $A = f^{-1}(U)$ and $B = f^{-1}(V)$ open in X. Then $X = A \cup B$ and $A \cap B = \varnothing$. This contradicts the connectedness of X. □

Example 2.65. Consider $p : \mathbb{E}^1 \to S^1, t \mapsto e^{2\pi i t}$. Since $\mathbb{E}^1$ is connected, S^1 is connected.

Corollary 2.66. *Connectedness is a topological property.*

Lemma 2.67. *Suppose X_0 is an open and closed subset of X and A is a connected subset of X. Then $A \subseteq X_0$ or $A \cap X = \varnothing$.*

Proof. Suppose not. Let $X_1 = X_0^c, U = A \cap X_0$ and $V = A \cap X_1$. Since X_0, X_1 are open subsets of X, U, V are open and nonempty, and $U \cap V = \varnothing$, which contradicts the connectedness of A. □

Proposition 2.68. *If X has a connected dense subset A, then X is connected.*

Proof. Let X_0 be an open and closed subset of X. By the lemma, $A \subseteq X_0$ or $A \cap X_0 = \varnothing$. If $A \subseteq X_0$, then $\overline{A} \subseteq X_0 \Rightarrow X \subseteq X_0 \Rightarrow X_0 = X$.

If $A \cap X_0 = \varnothing$, then $\overline{A} \cap X_0 = \varnothing \Rightarrow X \cap X_0 = \varnothing \Rightarrow X_0 = \varnothing$. Hence, X has only two open and closed subsets, X and $\varnothing$, which implies that X is connected. □

Recall Example 2.58, $X = \overline{A}$, then X is connected.

Corollary 2.69. *Z is a connected subset of X. Y is a subset of X such that $Z \subseteq Y \subseteq \overline{Z}$. Then Y is connected.*

Proof. Z is a connected dense subset of Y. □

Proposition 2.70. *$\mathscr{C}$ is a cover of X. If every member of $\mathscr{C}$ is connected and A is a connected subset of X such that the intersection of A and every member of $\mathscr{C}$ is nonempty, then X is connected.*

Proof. Let X_0 be an open and closed subset of X. If $X_0 \neq \varnothing$, since $\mathscr{C}$ is a cover of X, there exists $U \in \mathscr{C}$ such that $U \cap X_0 \neq \varnothing$. By Lemma 2.67, $U \subseteq X_0$ and $A \cap X_0 \neq \varnothing$. Again by Lemma 2.67, $A \subseteq X_0$ and for every $V \in \mathscr{C}$, $V \cap X_0 \neq \varnothing \Rightarrow V \subseteq X_0$. Hence, $X = X_0$ and X is connected. □

Example 2.71. $\mathbb{E}^n$ is connected.

Proof. Let $B_x = \{(x, y) | y \in \mathbb{E}^1\}$ for every $x \in \mathbb{E}^1$. Then $\{B_x | x \in \mathbb{E}^{n-1}\}$ is a connected cover of $\mathbb{E}^n$ and every member of $\mathscr{C}$ is connected. Let $A = \mathbb{E}^1 \times \{0\} \subseteq \mathbb{E}^n$, then A is connected and intersects every member of $\mathscr{C}$. By Proposition 2.70, $\mathbb{E}^n$ is connected. □

Example 2.72. S^n is connected.

Proposition 2.73. *X, Y are connected $\Longleftrightarrow$ $X \times Y$ is connected.*

Proof. $A = X \times \{y_0\}$ is connected. $\mathscr{C} = \{\{x\} \times Y | x \in X\}$ is a connected cover of $X \times Y$. The intersection of A and every member of $\mathscr{C}$ is nonempty. By Proposition 2.70 , $X \times Y$ is connected. □

Definition 2.74. A subset $X_0 \subseteq X$ is a **component** if X_0 is connected and is not properly contained in any connected subset of X.

Proposition 2.75. *Every connected nonempty subset A of X is contained in a unique component of X.*

Proof. Let $\mathscr{F} = \{F \subseteq X | F \text{ is connected}, A \cap F \neq \varnothing\}$ and $X_0 = \bigcup_{F \in \mathscr{F}} F$. $A \in \mathscr{F} \Rightarrow \mathscr{F}$ is nonempty and $A \subseteq X_0$. By Lemma 2.67, X_0 is connected.

If Y is connected and $X_0 \subseteq Y$, then $Y \cap A = A \neq \varnothing \Rightarrow Y \in \mathscr{F} \Rightarrow Y \subseteq X_0 \Rightarrow Y = X_0$. Hence, X_0 is a component of X. □

A component does not need to be closed. For example, consider $\{0\} \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup \{0\}$.

Proposition 2.76. *A component is closed.*

Proof. If A is a component of X and $A \neq \varnothing$, then $\overline{A}$ is connected and $A = \overline{A}$ is closed. □

Example 2.77. $T^2 = S^1 \times S^1$ is connected. A topological space with trivial topology is connected.

Example 2.78. $\mathbb{E}^1 \backslash S^0$ has three components $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.

Example 2.79. The components of $\mathbb{Q} \subset \mathbb{E}^1$ are the singleton sets.

2.5 Path Connectedness

Definition 2.80. A **path** in X is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0)$ is the **starting point** and $\gamma(1)$ is the **ending point**.

The **inverse** of path γ is $\overline{\gamma}$ such that $\overline{\gamma}(t) = \gamma(1 - t)$.

Given γ, η such that $\gamma(1) = \eta(0)$, the **concatenation** of γ and η is $\gamma\eta$ such that

$$(\gamma\eta)(t) = \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}] \\ \eta(2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}. \tag{2.1}$$

Definition 2.81. X is **path connected** if for every $x_0, x_1 \in X$, there exists a path from x_0 to x_1.

Example 2.82. Convex subsets of $\mathbb{E}^n$ are path connected.

Example 2.83. The topologist's sine curve in Example 2.58 is not path connected.

Proof. It suffices to show that if $a(0) \in B$, then $a(I) \subseteq B$. Denote $J^{-1} = a^{-1}(B)$, which is nonempty closed subset of I. We show that J^{-1} is also open in I.

For every $t \in J$, $a(t) \in B$. WLOG, assume $a(t) = (0, y)$ and $y \neq -1$. Define $U = \{(x, y) | y \neq -1\}$, which is an open neighborhood of $a(t)$. There exists a neighborhood W of t such that $a(W) \subseteq U$. WLOG, assume W is connected, then $a(W)$ is connected and is a subseteq of the component $B \backslash \{(0, -1)\}$ of U. Hence $W \subset J$ and t is an interior point of J.

Since I is connected, $J = I$ and $a(I) \subseteq B$. □

Proposition 2.84. *Path connectedness implies connectedness.*

Proof. For every $x_0 \in X$, which is a path connected space, denote $\mathscr{F} = \{\gamma([0, 1]) | \gamma(0) = x_0, \gamma : [0, 1] \to X \text{ is continuous }\}$. $\mathscr{F}$ is a connected cover of X, then by Proposition 2.70, X is connected. □

Proposition 2.85. *Connected open subsets of* $\mathbb{E}^n$ *are path connected.*

Proof. Let U be a connected open subset of $\mathbb{E}^n$. For $x_0 \in U$, denote $U_0 = \{x \in U |$ there exists a path in U from x_0 to $x\}$. U_0 is nonempty since $x_0 \in U_0$. For every $x \in U_0$, since U is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Since $x \in U_0$, there exists a path γ from x_0 to x. For every $y \in B(x, \varepsilon)$, let $\eta_y(t) = (1 - t)x + ty$ is the path from x to y. Then $\gamma\eta_y$ is a path from x_0 to y. Hence, $B(x, \varepsilon) \subseteq U_0$ and U_0 is open in U.

For $x \in \overline{U_0}$ (in U), there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$ and there exists $y \in B(x, \varepsilon) \cap U_0$. Let γ be the path in U_0 from x_0 to y and $\eta(t) = (1 - t)y + tx$ be the path in $B(x, \varepsilon) \subseteq U$ from y to x. Then $\gamma\eta$ is a path in U from x_0 to x. Hence, $x \in U_0$ and U_0 is closed in U. Since U is connected, $U_0 = U$. Therefore, U is path connected. □

Proposition 2.86. *The continuous image and product of a path connected space is path connected.*

Definition 2.87. A **path component** of X is a maximal path connected subset of X.

We define an equivalence relation on X: $x \sim y$ if there exists a path from x to y.

Proposition 2.88. *Path components are the equivalence classes of the relation $\sim$.*

Definition 2.89. X is **locally connected** if for every $x \in X$ and every neighborhood U of x, there exists a connected neighborhood V of x such that $V \subseteq U$.

Example 2.90. $X = \{(x, y) \in \mathbb{E}^2 | x \in \mathbb{Q} \text{ or } y = 0\}$ is locally connected.

Proof. Denote $H_+ = \{(x, y) \in X | y > 0\}$. For every $(x, y), y > 0$, $H_+ \cap X$ is an open neighborhood of (x, y). $H_+ \cap X$ does not contain a connected neighborhood in H_+ of (x, y). □

Proposition 2.91. *The components of a locally connected space are open.*

Proof. Let A be a component of X, which is locally connected. For every $x \in A$, there exists a connected neighborhood U of x, then $A \cap U$ is connected. Since A is a component, $U \subseteq A$ and x is an interior point of A. Hence, A is open. □

Definition 2.92. X is **locally path connected** if for every $x \in X$ and every neighborhood U of x, there exists a path connected neighborhood V of x such that $V \subseteq U$.

Example 2.93 (Comb Space)**.**

$$X = \{(x, y) \in [0, 1]^2 | x = 0 \text{ or } y = 0 \text{ or there exists } n \in \mathbb{N}^* \text{ such that } x = \frac{1}{n}\}.$$

Lemma 2.94. *If every $x \in X$ has a neighborhood U_x such that every point in U_x can be connected to x by a path in X, then*

1. *The path components of X are open and closed;*
2. *The components are path components.*

Proof. Every $x \in X$ is an interior point of the path component A containing x, so A is open. Since A is the complement of the union of other path components, A is closed.

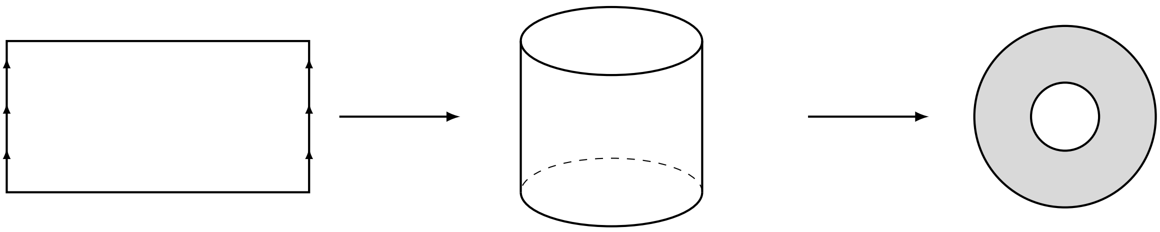
Let A be a path component and $B \supseteq A$ be the component containing A. Then A is an open and closed nonempty subset of B, hence $A = B$. □

As a corollary, we have the following proposition.

Proposition 2.95.
1. *A locally path connected connected space is path connected.*
2. *The path components of a locally path connected space are components and are open and closed.*

3 Quotient Spaces

3.1 Some Surfaces

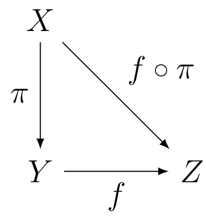


3.2 Quotient Spaces and Quotient Maps

Definition 3.1. X is a topological space. A collection P of nonempty subsets of X is called a **partition** of X if every element of X belongs to exactly one member of P and $\bigcup_{U \in P} U = X$.

Define Y such that the points in Y correspond to the members of P ($P \to Y$). Denote $\pi : X \to Y$ by $\pi(x)$ being the member of P containing x. Give Y the largest topology such that π is continuous, i.e. $O \subseteq Y$ is open $\iff \pi^{-1}(O)$ is open in X. This topology is called the **quotient topology**.

Proposition 3.2. *Y is a quotient space of X and $\pi : X \to Y$ is the natural map. Then for every topological space Z and function $f : Y \to Z$, f is continuous $\iff f \circ \pi$ is continuous.*



Proof. $\Longrightarrow$: Since π is continuous, $f \circ \pi$ is continuous.

$\Longleftarrow$: For $W \subseteq Z$ open, $(f \circ \pi)^{-1}(W) = U$ is open in X. Let $V = f^{-1}(W)$, then $(f \circ \pi)^{-1}(W) = \pi^{-1}(V)$. Since $U = \pi^{-1}(V)$ is open in X, V is open in Y. □

Definition 3.3. Define an equivalence relation $\sim$ on X by $x \sim y \iff$ there exists $U \in P$ such that $x, y \in U$. The quotient space determined by P is denoted by $X/\sim$ or X/P.

Definition 3.4. A continuous surjective map $f : X \to Y$ is called a **quotient map** if Y has the largest topology such that f is continuous, i.e. $U \subseteq Y$ is open $\iff f^{-1}(U)$ is open in X ($A \subseteq Y$ is closed $\iff f^{-1}(A)$ is closed in X).

Definition 3.5. Let $f : X \to Y$ be a continuous surjective map. $A \subseteq X$ is a **saturated set** if $A = f^{-1}(f(A))$.

Proposition 3.6. *For a quotient map $f : X \to Y$, define $x_1 \sim x_2 \iff f(x_1) = f(x_2)$, then $X/\sim \cong Y$.*

Proof. Let $p : X \to X/\sim$ be the natural map. Define $g : X/\sim \to Y$ such that $g([x]) = f(x)$, then g is well defined, bijective and continuous. By Proposition 3.2, since f is continuous, g is continuous. Since $p = g^{-1} \circ f$ is continuous, g^{-1} is continuous. Therefore, g is a homeomorphism. □

Proposition 3.7. *The composition of quotient maps is a quotient map.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quotient maps. $U \subseteq Z$ open $\iff g^{-1}(U)$ is open in $Y \iff f^{-1}(g^{-1}(U))$ is open in X. Hence, $g \circ f$ is a quotient map. □

Proposition 3.8. *Let $f : X \to Y$ and $g : X \to Z$ be quotient maps. If $\sim$ induced by f and g are the same, then $Y \cong Z$.*

Definition 3.9. $f : X \to Y$ is an **open map** if f maps open sets to open sets. f is a **closed map** if f maps closed sets to closed sets.

Proposition 3.10. *If $f : X \to Y$ is a continuous surjective open (closed) map, then f is a quotient map.*

Proof. For $f^{-1}(U)$ open (closed) in X, since f is an open (closed) map, U is open (closed) in Y. □

$p : X \times Y \to X$ is an open map, hence it is a quotient map. But p is not a closed map in general.

Example 3.11. For $\mathbb{E}^2 = \mathbb{E}^1 \times \mathbb{E}^1$, consider $\Gamma = \{(x, \arctan(x)) | x \in \mathbb{E}^1\}$ and $p((x, y)) = y$. Γ is closed in $\mathbb{E}^2$, but $p(\Gamma) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is open.

Example 3.12. For $p : \mathbb{E}^1 \to \mathbb{E}^1/\mathbb{Z}$ and $U = (-1, 1)$ open, $p^{-1}(p(U)) = U \cup \mathbb{Z}$ is open, but $p(U)$ is not open in $\mathbb{E}^1/\mathbb{Z}$. Hence, p is not an open map.

Proposition 3.13. *Continuous surjective maps from compact spaces to Hausdorff spaces are quotient maps.*

Proof. Let $f : X \to Y$ be continuous surjective, X be compact and Y be Hausdorff. For every closed set $E \subseteq X$, since X is compact, E is compact. Since f is continuous, $f(E)$ is compact in Y. Since Y is Hausdorff, $f(E)$ is closed in Y. □

For $Y \subseteq X$, define an equivalence relation on X: $a \sim b \iff a = b$ or $a, b \in Y$. The quotient space $X/\sim$ is denoted by X/Y.

Example 3.14. X is a topological space. A **cone** over X is the quotient space $CX = (X \times [0, 1])/(X \times \{1\})$. For a compact subset X of $\mathbb{E}^n$, denote $c_0 = (0, \cdots, 0, 1) \in \mathbb{E}^{n+1}$, then the **geometric cone** over X is $GX = \{t(x, 0) + (1 - t)c_0 | x \in X, t \in [0, 1]\}$.

Example 3.15. The quotient space of a Hausdorff space need not be Hausdorff. Consider $\mathbb{E}^1$ with an equivalence relation $x \sim y \iff x = y$ or $x, y < 0$. $X = \mathbb{E}^1/\sim$ is not T_2, because we can let $p : \mathbb{E}^1 \to \mathbb{E}^1/\sim$ and $Q = [-1]$, then any open neighborhood of $p(0)$ contains points in Q.

$f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are quotient maps. Define

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$$

such that $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Since f_1, f_2 are surjective, $f_1 \times f_2$ is surjective. But $f_1 \times f_2$ need not be a quotient map in general.

Example 3.16. $\mathbb{Q}, \mathbb{Z} \subset \mathbb{E}^1$ and $p : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a quotient map. Then $p \times 1 : \mathbb{Q} \times \mathbb{Q} \to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$ is not a quotient map.

Proof. Let $r_n = \frac{\sqrt{2}}{|n|+1}$, which is irrational and $r_n \to 0$. Let A_n is the interior of the quadrilateral with vertices $(n, r_n), (n + \frac{1}{2}, 0), (n + \frac{1}{2}, 2), (n + 1, r_{n+1})$. $\overline{A_n} \cap (\{n, n+1\} \times \mathbb{R}) = \{(n, r_n), (n + 1, r_{n+1})\}$. Let $A = \bigcup_{n \in \mathbb{Z}} A_n$ and $B = \overline{A} \cap (\mathbb{Q} \times \mathbb{Q})$. Then B is closed in $\mathbb{Q} \times \mathbb{Q}$ and saturated under $p \times 1$. But $(p \times 1)(B)$ is not closed, hence $p \times 1$ is not a quotient map. □

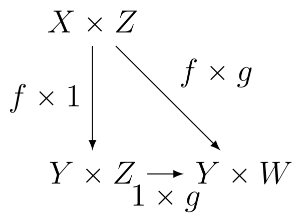
Theorem 3.17 (Whitehead)**.** *$p : X \to Y$ is a quotient map and Z is a locally compact Hausdorff space. Then $p \times 1 : X \times Z \to Y \times Z$ is a quotient map.*

Proof. $f = p \times 1$ is continuous and surjective. It suffices to show that for every subset $W \subseteq Y \times Z$, if $f^{-1}(W) \subseteq X \times Z$ is open, then W is open, i.e. $(y_0, z_0) \in W$ is an interior point.

Take $x_0 \in p^{-1}(y_0)$, then $(x_0, z_0) \in f^{-1}(W)$. There exists an neighborhood B of z_0 such that $\{x_0\} \times B \subseteq f^{-1}(W)$, then $y_0 \times B \subseteq W$. Since Z is locally compact Hausdorff, we can assume B is compact.

Let $U = \{x \in X | \{x\} \times B \subseteq f^{-1}(W)\}$ and $V = \{y \in Y | \{y\} \times B \subseteq W\}$, then $y_0 \in V$, $V \times B \subseteq W$ and $U = p^{-1}(V)$. Since B is compact and $f^{-1}(W)$ is open, for every $x \in U$, there exists an open neighborhood U_x of x such that $U_x \times B \subseteq f^{-1}(W)$. Hence $U_x \subseteq U$ and x is an interior point of U. Since p is a quotient map, V is open and (y_0, z_0) is an interior point of W. □

Corollary 3.18. *Suppose $f : X \to Y$ and $g : Z \to W$ are quotient maps. If Y and Z are locally compact Hausdorff, then $f \times g : X \times Z \to Y \times W$ is a quotient map.*



3.3 Classification of Closed Surfaces

Definition 3.19. $\mathbb{R}P^2 = S^2/\sim$, where $x \sim -x$ for every $x \in S^2$, is called the **projective plane** and $\mathbb{R}P^2 \cong X/\partial X$, where X is a closed unit disk.

Definition 3.20. A Hausdorff space X is a n-**dimensional (topological) manifold** or a n-**manifold** if every point $x \in X$ has an open neighborhood homeomorphic to $\mathbb{E}^n$ or $\mathbb{E}^n_+ = \{(x_1, x_2, \cdots, x_n) \in \mathbb{E}^n | x_n \geqslant 0\}$.

Definition 3.21. The **interior points** of a manifold X, denoted by $\mathrm{Int}(X)$ or X°, are the points having an open neighborhood homeomorphic to $\mathbb{E}^n$. If x has a neighborhood U homeomorphic to $\mathbb{E}^n_+$ such that $\varphi(x) = 0$, then x is a **boundary point** of X, denoted by ∂X.

Example 3.22. $\mathbb{E}^n$, $S^n \subset \mathbb{E}^{n+1}$ and $T^n = S^1 \times \cdots \times S^1$ are n-manifolds.

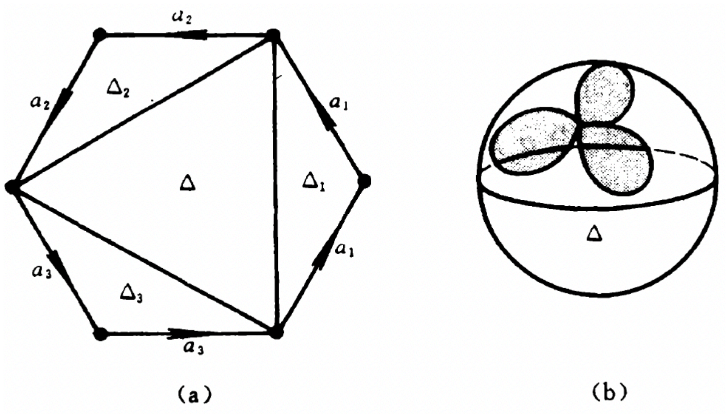
We have to admit that for $n \neq m$, $\mathbb{E}^n$ is not homeomorphic to $\mathbb{E}^m$ and we only consider compact manifolds (or its interior). If X is a n-manifold and $\partial X \neq \varnothing$, then ∂X is a $n+1$-manifold.

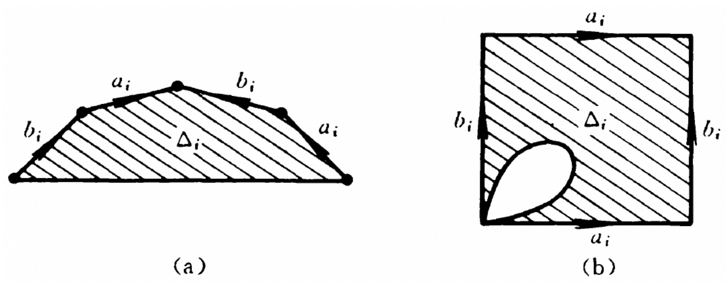
2-manifolds are called **surfaces**. For example, $S^2, \mathbb{D}^2, \mathbb{E}^2, T^2$, Möbius strip and Klein bottle are surfaces.

Theorem 3.23 (Classification of Closed Surfaces)**.** *Every closed surface (compact surface without boundaries) is homeomorphic to either S^2, nT^2 or mP^2 for some $n, m \in \mathbb{N}^*$. Any two surfaces in this list are not homeomorphic.*

Theorem 3.24 (Radó)**.** *Any closed surface can be obtained by identifying edges of a polygon in pairs.*

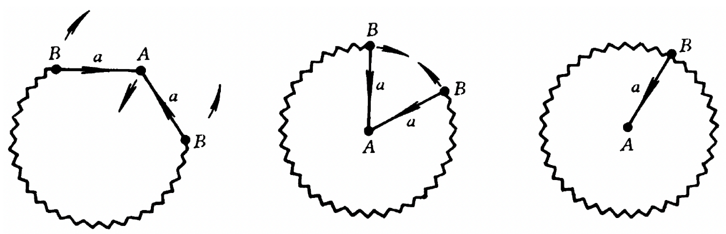
Hence, we have the standard presentation of closed surfaces $nT^2 = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$ and $mP^2 = a_1 a_1 \cdots a_m a_m$ as follows.



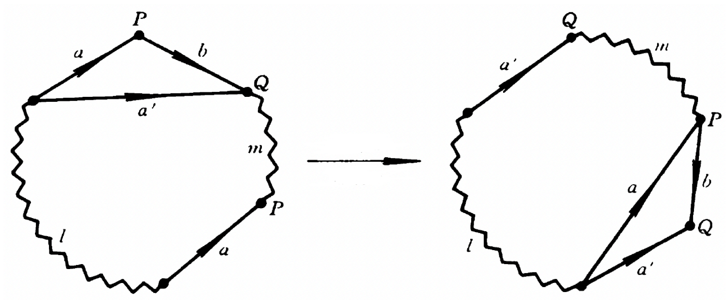


If two edges labeled by a are oriented in the same direction, a is called a **matching pair**; otherwise, it is called a **cross pair**. A **vertex category** is a set of vertices identified together after gluing the edges.

Proof. Step 1: Reduce to one vertex category. Suppose there are more than one category, then pick one category P. If P has only one preimage vertex, then we can do the following operation.

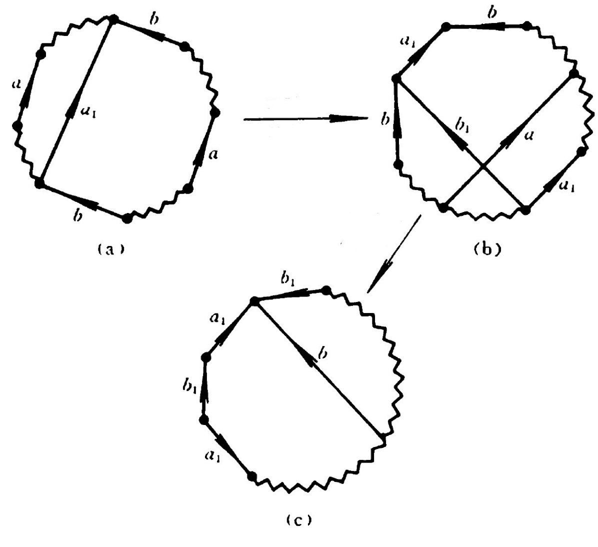


If P has more than one preimage vertex, pick one preimage vertices such that the adjoint vertex is the preimage of another category. Do the following operation to reduce P's preimage vertices by 1.

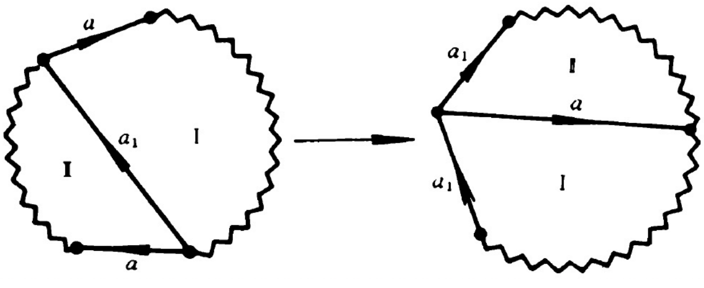


Repeat the process until there is only one vertex category.

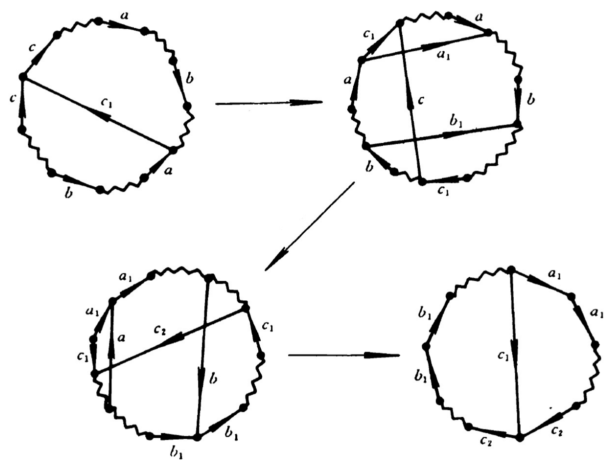
Step 2: Make the order of edges standard. It's easy to see that after Step 1, cross pairs must be disjoint and there must be another pair alternating with them. If there is no matching pair, do the following operation to make them adjoint. And in the end, we can get $a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$.



If there are matching pairs, do the following operation to make them adjoint.



If there still are cross pairs, do the following operation to convert them into matching pairs.



□

Therefore, if there are matching pairs, the surface is homeomorphic to mP^2; otherwise, it is homeomorphic to nT^2. If there are l edges and k vertex categories, then the edges number in standard representation is $l - 2k + 2$.

Theorem 3.25 (Banchoff). *P^2 cannot be embedded in $\mathbb{E}^3$. If P^2 is immersed in $\mathbb{E}^3$ by $f : P^2 \to \mathbb{E}^3$, then there must be at least one triple point.*

4 Homotopy and Fundamental Group

Fundamental group is an important tool to study topological spaces. It can distinguish some topological spaces which are not homeomorphic. But actually, by the following theorem, in some cases, it is not useful.

Theorem 4.1. *There is no algorithm to decide whether two finite presentation groups are isomorphic.*

4.1 Homotopy

Definition 4.2. $f, g : X \to Y$ are **homotopic** if there exists a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. Denote $f \simeq g$ and H is called a **homotopy** from f to g.

Example 4.3. $f, g : X \to \mathbb{E}^n$ are continuous. Then $f \cup g : X \times \{0,1\} \to \mathbb{E}^n$ can be extended to $H : X \times [0,1] \to \mathbb{E}^n$ by $H(x,t) = (1-t)f(x) + tg(x)$. Hence, $f \simeq g$ and H is called a **linear homotopy**.

Actually, any two continuous maps from a space to a convex subset of $\mathbb{E}^n$ are homotopic.

Example 4.4. $f, g : X \to S^n$ and $f(x) \neq -g(x)$ for every $x \in X$. Then the homotopy is given by $H : X \times [0,1] \to S^n$

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

Proposition 4.5. *Homotopy is an equivalence relation on $C(X,Y)$.*

Proof. Reflexivity: $f \simeq f$ by $H(x,t) = f(x)$.

Symmetry: If $f \simeq g$ by H, then $g \simeq f$ by $G(x,t) = H(x, 1-t)$.

Transitivity: If $f \simeq g$ by H and $g \simeq k$ by G, then $f \simeq k$ by

$$F(x,t) = \begin{cases} H(x,2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ G(x, 2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

□

Definition 4.6. The equivalence classes of homotopy are called **map classes**. The set of map classes is denoted by $[X, Y]$.

$[X, \mathbb{E}^n]$ has only one element.

Example 4.7. If $X = \{x_0\}$, then $[X,Y]$ is in one-to-one correspondence with Y. Two maps f, g are homotopic $\iff$ there exists a path from $f(x_0)$ to $g(x_0)$ in Y. Therefore, $[X,Y]$ corresponds to the path components of Y.

Proposition 4.8. *If $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$, then the homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$ is given by

$$H(x,t) = G(F(x,t),t).$$

$H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0(f_0(x))$ and $H(x,1) = G(F(x,1),1) = G(f_1(x),1) = g_1(f_1(x))$. □

Definition 4.9. f is a **null homotopy** if f is homotopic to a constant map.

If Y is a convex subset of $\mathbb{E}^n$, then every map $f : X \to Y$ is null homotopic.

Definition 4.10. $A \subset X$ and $f, g \in C(X, Y)$ such that $f|_A = g|_A$. f and g are **homotopic relative to** A if there exists a homotopy H from f to g such that for every $a \in A$ and $t \in [0, 1]$, $H(a, t) = f(a) = g(a)$, denoted by $f \simeq g$ rel A.

Example 4.11. Y is a convex subset of $\mathbb{E}^n$ and $f, g \in C(X, Y)$. Let $H(x, t) = (1 - t)f(x) + tg(x)$, then if $f|_A = g|_A$, we have $f \simeq g$ rel A.

Relative homotopy is also an extension of maps.

Example 4.12. $X = S^1$ and $x_0 \in S^1$. Let $Y = S^1 \times [0, 1]/\{(x_0, 0), (x_0, 1)\}$.

Proposition 4.13. *Relative homotopy is an equivalence relation and remains with maps.*

4.2 Definition of Fundamental Group

X is a topological space and take $x_0 \in X$, called the **base point**. A **loop** based at x_0 is a continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1) = x_0$. Denote $\mathscr{L}(X, x_0)$ the set of all loops based at x_0.

Define the **product** of two path categories $\langle\alpha\rangle, \langle\beta\rangle$ by $\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha \cdot \beta\rangle$.

Definition 4.14. The **fundamental group** of X based at x_0 is the set of all path categories of loops based at x_0 with the product defined above, denoted by $\pi_1(X, x_0)$.

The identity element is e_{x_0} defined by $e_{x_0}(t) = x_0$ for every $t \in [0, 1]$. The inverse of $\langle\alpha\rangle$ is $\langle\overline{\alpha}\rangle$, where $\overline{\alpha}(t) = \alpha(1 - t)$ for every $t \in [0, 1]$. Then it is easy to check that $\pi_1(X, x_0)$ is a group.

Example 4.15. X is a convex subset of $\mathbb{E}^n$ and $x_0 \in X$. Any two loops based at x_0 are relatively homotopic, hence $\pi_1(X, x_0) = \{\langle e\rangle\}$, i.e. the fundamental group is trivial.

Definition 4.16. A space X is **simply connected** if X is path-connected and its fundamental group is trivial.

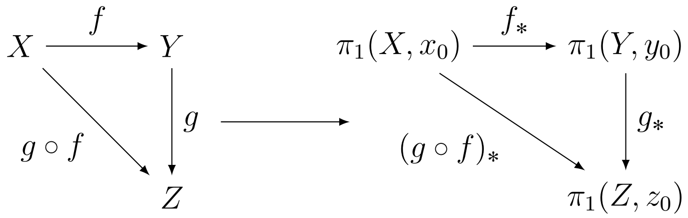
$f : (X, x_0) \to (Y, y_0)$ is a continuous map such that $f(x_0) = y_0$. For any loop α based at x_0, $f \circ \alpha$ is a loop based at y_0. Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$. Then f_* is a homomorphism.

Definition 4.17. $f : X \to Y$ is a continuous map such that $f(x_0) = y_0$. Then we call homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ the **induced fundamental group homomorphism** by f.

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{\pi_1} & \pi_1(X, x_0) \\ \Big\downarrow f & \longrightarrow & \Big\downarrow f_* \\ (Y, y_0) & \longrightarrow & \pi_1(Y, y_0) \end{array}$$

Proposition 4.18. *$f: X \to Y$ and $g: Y \to Z$ are continuous maps such that $f(x_0) = y_0$ and $g(y_0) = z_0$. Then*

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0).$$



Proof. Suppose $\alpha \in \mathscr{L}(X, x_0)$, then

$$(g \circ f)_*(\langle \alpha \rangle) = \langle (g \circ f) \circ \alpha \rangle = (\langle g \circ (f \circ \alpha) \rangle) = g_* \circ f_*(\langle \alpha \rangle).$$

□

$1_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity homomorphism.

Theorem 4.19. *$f : X \to Y$ is a homeomorphism such that $f(x_0) = y_0$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Denote $g = f^{-1} : (Y, y_0) \to (X, x_0)$, then $g \circ f = 1_X$ and $f \circ g = 1_Y$. By Proposition 4.18, we have

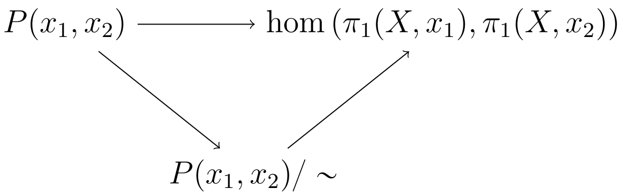
$$g_* \circ f_* = 1 : \pi_1(X, x_0) \to \pi_1(X, x_0), \quad f_* \circ g_* = 1 : \pi_1(Y, y_0) \to \pi_1(Y, y_0).$$

Hence, f_* is an isomorphism. □

By Theorem 4.19, if $(X, x_0) \cong (Y, y_0)$, we have $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, i.e. fundamental group is a topological invariant.

Now we study the relation between fundamental groups and base points. $x_1, x_2 \in X$ and suppose there exists a path ω from x_1 to x_2. For any $\langle \alpha \rangle \in \pi_1(X, x_1)$, $\langle \overline{\omega} \cdot \alpha \cdot \omega \rangle \in \pi_1(X, x_2)$. Then we define $\omega_\# : \pi_1(X, x_1) \to \pi_1(X, x_2)$ by $\omega_\#(\langle \alpha \rangle) = \langle \overline{\omega} \cdot \alpha \cdot \omega \rangle$.

If ω and ω' are homotopic relative to $\{0, 1\}$, then $\overline{\omega} \cdot \alpha \cdot \omega$ and $\overline{\omega'} \cdot \alpha \cdot \omega'$ are homotopic relative to $\{0, 1\}$. Hence, $\omega_\# = \omega'_\#$.



Proposition 4.20. *ω, ω' are paths from x_1 to x_2 and from x_2 to x_3, respectively. Then*

$$(\omega\omega')_\# = \omega'_\# \circ \omega_\# : \pi_1(X, x_1) \to \pi_1(X, x_3).$$

Proof.

$$(\omega\omega')_\# (\langle \alpha \rangle) = \langle \overline{\omega\omega'} \cdot \alpha \cdot (\omega\omega') \rangle = \langle \overline{\omega'} \cdot (\overline{\omega} \cdot \alpha \cdot \omega) \cdot \omega' \rangle = \omega'_\# \circ \omega_\#(\langle \alpha \rangle).$$

□

Proposition 4.21. *ω is a path category from x_1 to x_2. Then $\omega_\# : \pi_1(X, x_1) \to \pi_1(X, x_2)$ is an isomorphism. If $\omega \in \pi_1(X, x_1)$, then $\omega_\#$ is an inner isomorphism.*

Proof. First, $\omega_\#$ is a homomorphism. For $\overline{\omega} : \pi_1(X, x_2) \to \pi_1(X, x_1)$, we have $\overline{\omega}_\# \circ \omega_\# = 1 : \pi_1(X, x_1) \to \pi_1(X, x_1)$ and $\omega_\# \circ \overline{\omega}_\# = 1 : \pi_1(X, x_2) \to \pi_1(X, x_2)$. Hence, $\omega_\#$ is injective and $\overline{\omega}_\#$ is surjective. Similarly, $\omega_\#$ is surjective and $\overline{\omega}_\#$ is injective. Therefore, $\omega_\#$ is an isomorphism. □

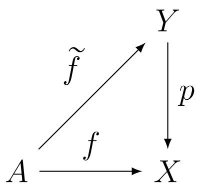
If X is path-connected, then $\pi_1(X, x_1) \cong \pi_1(X, x_2)$. Hence the fundamental group is independent of the base point up to isomorphism, denoted $\pi_1(X)$.

4.3 Fundamental Group of S^n

Consider $S^1 = \{z \in \mathbb{C}^2 | |z| = 1\}$. Define $p : \mathbb{E}^1 \to S^1$ by $p(t) = e^{2\pi i t}$. p is a homeomorphism locally: for any $t \in \mathbb{E}^1$, $p(t - \frac{1}{2}, t + \frac{1}{2}) = S^1 \backslash \{p(t + \frac{1}{2})\}$.

Theorem 4.22. *Denote ω_n is a closed path and define $\omega_n : [0, 1] \to S^1$ by $\omega_n(t) = e^{2\pi i n t}$. Then $\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ defined by $\Phi(n) = \langle \omega_n \rangle$ is an isomorphism.*

Definition 4.23. $p : Y \to X$ and $f : A \to X$ are continuous. If there exists $\widetilde{f} : A \to Y$ such that $f = p \circ \widetilde{f}$, then $\widetilde{f}$ is called a **lift** of f under p.



Lemma 4.24. *Given $\alpha : [0, 1] \to S^1$ with $\alpha(0) = x_0 \in S^1$. For any $\widetilde{x_0} \in p^{-1}(x_0)$, there exists a unique lift $\widetilde{\alpha} : [0, 1] \to \mathbb{E}^1$ such that $\widetilde{\alpha}(0) = \widetilde{x_0}$.*

Lemma 4.25. *Any homotopy of*

Now we consider the higher dimensional sphere S^n for $n \geqslant 2$.

Theorem 4.26. *Assume $X = U \cup V$, where U, V are open and simply connected. If $U \cap V$ is path-connected, then $\pi_1(X) \cong \{e\}$.*

Proof. For any closed path $\alpha : [0, 1] \to X$ with $\alpha(0) = x_0 \in U \cap V$, by Lebesgue Lemma, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\alpha([t_i, t_{i+1}]) \subseteq U$ or V. WLOG, assume $\alpha([t_i, t_{i+1}])$ and $\alpha([t_{i+1}, t_{i+2}])$ do not lie in the same set, then $\alpha(t_i) \in U \cap V$. Since $U \cap V$ is path-connected, there exists a path $w_i : [0, 1] \to U \cap V$ such that $w_i(0) = \alpha(t_i)$ and $w_i(1) = x_0$. Denote $\alpha_i(s) = a(t_i + s(t_{i+1} - t_i))$. Then $\langle \alpha \rangle = \langle \alpha_0 \cdot \alpha_1 \cdot \cdots \cdot \alpha_{m-1} \rangle = \langle \alpha_0 \cdot w_1 \cdot \overline{w_1} \cdot \alpha_1 \cdots \rangle = \langle \alpha_0 \cdot w_1 \rangle \langle \overline{w_1} \cdot \alpha_1 \cdots w_2 \rangle$. □

For S^n, denote the north pole p and the south pole q. SInce $(S^n \backslash \{p\}) \cap (S^n \backslash \{q\}) = S^n \backslash \{p, q\}$ is path-connected for $n \geqslant 2$ and both $S^n \backslash \{p\}$ and $S^n \backslash \{q\}$ are homeomorphic to $\mathbb{E}^n$, by the above theorem, S^n is simply connected for $n \geqslant 2$.

Theorem 4.27. *If X and Y are path-connected, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof. Take $x_0 \in X$, $y_0 \in Y$ and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. Define $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\varphi(\langle\gamma\rangle) = (\langle p_1 \circ \gamma\rangle, \langle p_2 \circ \gamma\rangle)$, which is a morphism.

φ is surjective: For any $(\langle\alpha\rangle, \langle\beta\rangle) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$, define $\gamma : [0,1] \to X \times Y$ by $\gamma(t) = (\alpha(t), \beta(t))$, then $\varphi(\langle\gamma\rangle) = (\langle\alpha\rangle, \langle\beta\rangle)$.

φ is injective: If $\varphi(\langle\gamma\rangle) = (e_X, e_Y)$, then $p_1 \circ \gamma \simeq c_{x_0}$ and $p_2 \circ \gamma \simeq c_{y_0}$. Let $H_1 : p_1 \circ \gamma \simeq c_{x_0}$ and $H_2 : p_2 \circ \gamma \simeq c_{y_0}$, then the homotopy from γ to $c_{(x_0,y_0)}$ is given by

$$H(t,s) = (H_1(t,s), H_2(t,s)).$$

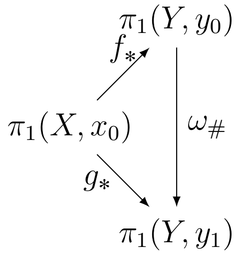
□

Since $T^2 \cong S^1 \times S^1$, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, hence $T^2 \ncong S^2$. For any n, $\pi_1(T^n) \cong \mathbb{Z}^n$.

4.4 Homotopy Invariance of Fundamental Group

Consider $f, g : X \to Y$. If $f \simeq g$, denote $H : X \times [0,1] \to Y$. For $x_0 \in X$, $y_0 = f_(x_0)$ and $y_1 = g(x_0)$. There are induced homomorphisms $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$. Define $h_t : X \to Y$ by $h_t(x) = H(x,t)$ and $\omega(t) = x_t = H(x_0, t)$, then ω is a path from y_0 to y_1.

Proposition 4.28. $g_* = \omega_\# \circ f_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$.



Definition 4.29. X, Y are topological spaces. Iff there are $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$, then X and Y are **homotopy equivalent**, denoted by $X \simeq Y$. f and g are called **homotopy equivalences** and they are **homotopy inverses** of each other.

Proposition 4.30. *Homotopy equivalence is an equivalence relation on topological spaces.*

Proof. Suppose $f : X \to Y$ and $g : Y \to X$ are homotopy equivalences and $u : Y \to Z$ and $v : Z \to Y$ are homotopy equivalences. It suffices to show that $u \circ f : X \to Z$ and $g \circ v : Z \to X$ are homotopy equivalences.

$(g \circ v) \circ (u \circ f) = g \circ (v \circ u) \circ f \simeq g \circ 1_Y \circ f \simeq g \circ f \simeq 1_X$. □

Example 4.31. Homeomorphisms are homotopy equivalences. If $f : X \to Y$ is a homeomorphism, then all the homotopy inverses of f form a class of maps from Y to X.

Example 4.32. $p : X \times [0,1] \to X$ and $i : X \to X \times [0,1]$ defined by $i(x) = (x, 0)$. Then $p \circ i = 1_X$ and $i \circ p(x,t) = (x, 0)$. Define $H(x,s,t) = (x, st)$.

Example 4.33. Convex subsets of $\mathbb{E}^n$ are homotopy equivalent to a point.

Definition 4.34. If X is homotopy equivalent to a point, then X is **contractible**.

Proposition 4.35. *If $f : X \simeq Y$ and $y_0 = f(x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Hence if X and Y are homotopy equivalent and path connected, then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. Supppose $g : Y \to X$ is a homotopy inverse of f and $x_1 = g(y_0)$. Assume $g \circ f \simeq 1_X$ by H and $\omega(t) = H(x_0, t)$ is a path from x_0 to x_1. Then $g_* \circ f_* = \omega_\# \circ (1_x)_* = \omega_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism. Hence $g_* \circ f_*$ is an isomorphism, which implies that f_* is injective and g_* is surjective. Similarly, $f_* \circ g_*$ is an isomorphism, which implies that f_* is surjective and g_* is injective. Therefore, f_* and g_* are isomorphisms. □

The fundamental group of a contractible space is trivial, hence simply connected.

Proposition 4.36. *X is contractible $\iff$ 1_X is homotopic to a constant map $\implies$ every map from Y to X is null homotopic.*

Definition 4.37. $A \subset X$ is a **retract** of X if there exists a map $r : X \to A$ such that $r|_A = 1_A$. r is called a **retraction**.

A is a **deformation retract** of X if there exists a homotopy $H : X \times [0, 1] \to X$ such that $H(x, 0) = x$, $H(x, 1) \in A$ for every $x \in X$ and $H(a, 1) = a$ for every $a \in A$. H is called a **deformation retraction**.

A deformation retraction is a **strong deformation retraction** if $H(a, t) = a$ for every $a \in A$ and $t \in [0, 1]$.

Example 4.38. $T^2 = S^1 \times S^1$ and $A = S^1 \times \{1\}$. $r : T^2 \to A$ defined by $r(a, b) = (a, 1)$ is a retraction. $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^1) \cong \mathbb{Z} \implies T^2 \not\simeq S^1$.

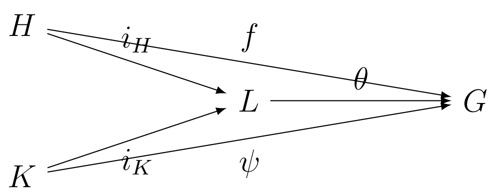
4.5

Definition 4.39. For two groups G and H, the **free product**

$$G * H = \{g_1 h_1 g_2 h_2 \cdots g_n h_n | g_i \in G, h_i \in H, n \in \mathbb{N}\}.$$

Theorem 4.40 (Universal Property of Free Product)**.** *Given two groups H and K, there exists a group L and homomorphisms $i_H : H \to L$ and $i_K : K \to L$ such that for any group G and homomorphisms $\varphi : H \to G$ and $\psi : K \to G$, there exists a unique homomorphism $\theta : L \to G$ such that $\theta \circ i_H = \varphi$ and $\theta \circ i_K = \psi$.*

*L is unique up to isomorphism and is called the **free product** of H and K, denoted by $H * K$.*

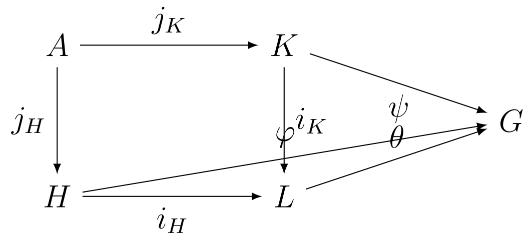


Theorem 4.41. *Given groups H, K and A and homomorphisms $j_H : A \to H$ and $j_K : A \to K$, there exists a group L and homomorphisms $i_H : H \to L$ and $i_K : K \to L$ such that*

1. $i_H \circ j_H = i_K \circ j_K$,

2. *For any group G and homomorphisms $\varphi : H \to G$ and $\psi : K \to G$, if $\varphi \circ j_H = \psi \circ j_K$, then there exists a unique homomorphism $\theta : L \to G$ such that $\theta \circ i_H = \varphi$ and $\theta \circ i_K = \psi$.*

L is unique up to isomorphism and is called the ***amalgamated free product*** *of H and K over A, denoted by $H *_A K$.*



It is easy to see that $H *_{\langle e \rangle} K \cong H * K$.

For group $A \subset G$, denote $[A]$ the smallest normal subgroup of G containing A.

Let $N = [\{j_H(a)j_K(a)^{-1} | a \in A\}]$, then $H *_A K \cong (H * K)/N$.

Generally, $\pi_1(X_1) * \pi_1(X_2) \to \pi_1(X_1 \cup X_2)$ is not an injection.

Theorem 4.42 (Van-Kampen Theorem)**.** *$X = X_1 \cup X_2$, where X_1, X_2 are open and path-connected and $X_0 = X_1 \cap X_2 \neq \varnothing$ is path-connected. For any $x_0 \in X_0$,*

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) *_{\pi_1(X_0, x_0)} \pi_1(X_2, x_0) \cong \frac{\pi_1(X_1, x_0) * \pi_1(X_2, x_0)}{[\{j_1(\alpha)j_2(\alpha)^{-1} | \alpha \in \pi_1(X_0, x_0)\}]}, \tag{4.1}$$

where $j_1 : \pi_1(X_0, x_0) \to \pi_1(X_1, x_0)$ and $j_2 : \pi_1(X_0, x_0) \to \pi_1(X_2, x_0)$ are induced by inclusion maps.

Theorem 4.43. *When X_1 and X_2 are closed, the Van-Kampen Theorem still holds if X_0 is a strong deformation retract of an open neighborhood in X.*

If $X_1 \cap X_2$ is simply connected, then $\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0)$.

If X_1 (or X_2) is simply connected, then $\pi_1(X, x_0) \cong \pi_1(X_2, x_0)/[\Im(i_2)_*]$.

Example 4.44. $S^2 = (S^2 \backslash \{N\}) \cup (S^2 \backslash \{S\})$, where N and S are the north pole and the south pole, respectively. Both $S^2 \backslash \{N\}$ and $S^2 \backslash \{S\}$ are homeomorphic to $\mathbb{E}^2$, hence simply connected. Their intersection is $S^2 \backslash \{N, S\} \cong S^1 \times \mathbb{R}$, which is homotopy equivalent to S^1. By Van-Kampen Theorem,

$$\pi_1(S^2) \cong \pi_1(S^2 \backslash \{N\}) *_{\pi_1(S^2 \backslash \{N,S\})} \pi_1(S^2 \backslash \{S\}) \cong \{e\} *_{\mathbb{Z}} \{e\} \cong \{e\}.$$

Proof. $\mathscr{F} = \{X_\alpha | \alpha \in \Lambda\}$ is a path-connected open cover of X. Assume $\bigcup_{\alpha \in \Lambda} X_\alpha \neq \varnothing$ and $x_0 \in X_{\alpha_0}$. For any $\alpha \in \bigcup_{\alpha \in \Lambda} X_\alpha$. $j_\alpha : \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0)$ is induced by inclusion map and $i_{\alpha\beta} : \pi_1(X_\alpha \cap X_\beta, x_0) \to \pi_1(X_\alpha, x_0)$ is induced by inclusion map. j_α can be extended to $\Phi : *_{\alpha \in \Lambda} \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0)$.

If for any $\alpha, \beta \in \Lambda$, $X_\alpha \cap X_\beta$ is path-connected, then Φ is surjective.

$\omega : [0, 1] \to X$ is a closed path based on x_0. By Lebesgue Lemma, there exists a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ such that for any j, there exists $X_j \in \{X_\alpha | \alpha \in \Lambda\}$ such that $\omega([s_j, s_{j+1}]) \subseteq X_j$. Since $X_{j-1} \cap X_j$ is path-connected, there exists a path $\sigma_j \in X_{j-1} \cap X_j$ such that connecting $\omega(s_j)$ and x_0. Denote $\omega_j(t) = \omega((1-t)s_j + ts_{j+1})$. Then $\omega \simeq (\omega_0 \circ \sigma_1)(\overline{\sigma}_1 \circ \omega_2 \sigma_2) \cdots (\overline{\sigma}_{n-1} \circ \omega_{n-1})$. Hence $\langle \omega \rangle = \langle \omega_0 \circ \sigma_1 \rangle \langle \overline{\sigma}_1 \circ \omega_2 \sigma_2 \rangle \cdots \langle \overline{\sigma}_{n-1} \circ \omega_{n-1} \rangle \Longrightarrow \langle \omega \rangle \in \mathrm{Im}(\Phi)$. Hence Φ is surjective.

For any $\alpha, \beta, \gamma \in \Lambda$, $X_\alpha \cap X_\beta \cap X_\gamma$ is path-connectedm then $\ker \Phi = N = [\{i_{\alpha\beta}(\omega) i^{-1}_{\beta\aleph}(\omega_1) | \omega \in \pi_1(X_\alpha \cap X_\beta \cap X_\gamma, x_0), \alpha, \beta \in \Lambda\}]$.

Two decompositions of $\langle\omega\rangle$ are equivalent if they can be transformed into each other by following .

1. If $\langle\omega_i\rangle, \langle\omega_{i+1}\rangle \in \pi_1(X_{\alpha_j})$, then replace $\langle\omega_j\rangle, \langle\omega_{j+1}\rangle$ by $\langle\omega_j\omega_{j+}\rangle$
2. If $\omega_i \in X_\alpha \cap X_\beta$, replace $\langle\omega_i\rangle \in \pi_1(X_\alpha)$ by $\langle\omega_i\rangle \in \pi_1(X_\beta)$.

Claim: Any two decompositions of $\langle\omega\rangle \in \pi_1(X)$ are equivalent.

If $\langle\omega_1\rangle \cdots \langle\omega_n\rangle$ and $\langle\omega'_1\rangle \cdots \langle\omega'_m\rangle$ are two decompositions of $\langle\omega\rangle$, then $\omega_1\omega_2\cdots\omega_n$ and $\omega'_1\omega'_2\cdots\omega'_m$ are homotopic relative to $\{0, 1\}$. $F : [0, 1] \times [0, 1]$ is the homotopy between them. By Lebesgue Lemma, there exists a partition $0 = s_0 < s_1 < \cdots < s_k = 1$ and $0 = t_0 < t_1 < \cdots < t_l = 1 (l \geqslant 3)$ such that for any rectangle $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$, there exists $\alpha \in \Lambda$ such that $F([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subseteq X_\alpha$.

Modify the partition $\{t_j\}$ such that for $t = 0, 1$, the partition are exactly $\omega_1\omega_2\cdots\omega_n$ and $\omega'_1\omega'_2\cdots\omega'_m$.

If $F([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subseteq X_\alpha$, we can map some open neighborhood of R_{ij} to X_α. We can disturb it such that every vertex

We label the rectangles from left to right and from bottom to top, by $R_1, R_2, \cdots, R_{kl}$. Take X_r such that $F(R_r) \subset X_r$. For $1 \leqslant r \leqslant kl$, take σ_r is the path separating $R_1 \cdots R_r$ and $R_{r+1} \cdots R_{kl}$ from left to right. $\omega_1 \cdots \omega_n \simeq \sigma_0$ and $\omega'_1 \cdots \omega'_m \simeq \sigma_{kl}$.

For every vertex V, if $F(V) \neq x_0$, take a path n_V connecting $F(V)$ and x_0 such that

1. n_V is in X_r corresponding to the rectangle R_r containing V.
2. If V is $[0, 1] \times \{0\}$, V is on R_r and R_{r+1}. If $F(V) \neq x_0$

There is an isomorphism $\Phi : *_{\alpha\in\Lambda}\pi_1(X_\alpha, x_0)/N \to \pi_1(X, x_0)$, where $N = [\{i_{\alpha\beta}(\gamma) i^{-1}_{\beta\alpha}(\gamma) | \gamma \in \pi_1(X_\alpha \cap X_\beta, x_0), \alpha, \beta \in \Lambda\}]$. □

$\pi_1(nT^2) \cong \langle \alpha_1, \beta_1, \cdots \alpha_n, \beta_n | [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n] = 1\rangle$ and $\pi_1(mP^2) \cong \langle \alpha_1, \alpha_2, \cdots, \alpha_m | \alpha_1^2\alpha_2^2 \cdots$ $1\rangle$.

Definition 4.45. $G/[G, G]$ is an abelian group called the **abelianization** of G, denoted by G^{abel}.

Theorem 4.46. *Every finite generated abelian group is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_m}$, where q_i are powers of primes. G is finite iff $n = 0$ and $q_1, \cdots, q_m$ are unique up to order.*

Corollary 4.47. *$\pi_1(nT^2) \ncong \pi_1(mP^2)$.*