One-Dimensional Quantum Many-Body Problems

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Abstract

This note is based on C. N. Yang's work on one-dimensional quantum many-body problems [2].

Contents

1 Background

The first postulate of quantum mechanics states that the state of a quantum mechanical system is completely specified by a wave function $\psi(x,t)$ in Hilbert space. The second postulate of quantum mechanics states that the time evolution of the wave function is governed by the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = H\psi(\boldsymbol{r},t), \tag{1.1}$$

where H is the Hamiltonian operator and $H = -\nabla^2 + V(\boldsymbol{r})$ in the position representation.

To solve the Schrödinger equation, we can solve the eigenvalue equation

$$H\psi = E\psi, \tag{1.2}$$

and get a complete basis ψ_n and its corresponding eigenvalue E_n. Hence, we can expand the wave function with the basis:

$$\psi(t) = \sum_n c_n \psi_n e^{-iE_n t}. \tag{1.3}$$

And we can determine the coefficients c_n by the initial condition $\psi(0)$. Therefore, the main task is to solve the eigenvalue equation.

However, in general this is not that easy. Here are some simple examples. Sutherland introduced and studied the eigenfunctions of

$$H_N = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{j<k}\frac{1}{\sin^2\frac{1}{2}(x_j - x_k)}, \tag{1.4}$$

and the eigenfunctions are given by the Jack polynomials. Specially, when $N = 1$, the eigenfunctions are given by the Gegenbauer polynomials.

In this note, we will study the 'delta-function' interaction model first solved by Lieb and Liniger for bosonic cases, and Yang gave a solution for the general cases. The Hamiltonian of the system is given by

$$H = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}\delta(x_i - x_j). \tag{1.5}$$

2 Single Particle Version

First we consider the quantum system of a single particle in a 'delta-function' potential. The Hamiltonian is given by

$$H = -\frac{\partial^2}{\partial x^2} + 2c\delta(x) \tag{2.1}$$

When $x \neq 0$, the wavefunction satisfies

$$-\frac{\partial^2\psi}{\partial x^2} = E\psi.$$

And the general solution is given by

$$\psi(x) = \begin{cases} e^{-ikx} + S_R(k)e^{ikx}, & x > 0, \\ S_T(k)e^{-ikx}, & x < 0, \end{cases}, \quad k = \sqrt{E}. \tag{2.2}$$

Consider the boundary conditions at $x = 0$. First integrate $H\psi = E\psi$ twice, from $x = -\epsilon$ to $x = y$ and $y = -\epsilon$ to $y = \epsilon$, we have

$$\psi(+0) = \psi(-0). \tag{2.3}$$

By integrating the eigenvalue equation once, from $x = -\epsilon$ to $x = \epsilon$, we have

$$\psi'(+0) - \psi'(-0) = 2c\psi(0). \tag{2.4}$$

Hence, we have

$$S_R(k) = \frac{c}{ik - c}, \quad S_T(k) = \frac{ik}{ik - c}. \tag{2.5}$$

3 Yang-Baxter Equation

The Hamiltonian of the system is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j), \tag{3.1}$$

where c is a positive constant characterizing the interaction strength between particles.

Let $p_1, \cdots, p_N$ be a set of unequal numbers. We make the assumption that the particle momenta are conserved in collisions, thus ψ_Q in each region $\Delta_Q : 0 < x_{Q(1)} < \cdots < x_{Q(N)} < L$ is a linear combination of plane waves $e^{i(k_{P(1)}x_{Q(1)} + \cdots + k_{P(N)}x_{Q(N)})}$. By Bethe's ansatz, we can write the wavefunction as

$$\psi_Q = \sum_P [Q, P] e^{i(k_{P(1)}x_{Q(1)} + \cdots + k_{P(N)}x_{Q(N)})}, \quad P \in S_N. \tag{3.2}$$

All the $N! \times N! [Q, P]$ can be arranged into a matrix and denote its columns by

$$\xi_P = \begin{pmatrix} [Q_1, P] \\ [Q_2, P] \\ \vdots \\ [Q_{N!}, P] \end{pmatrix}.$$

Following the single particle case and denoting $x = x_j - x_k, y = x_j + x_k$, we have boundary conditions at $x_j = x_k$:

$$\psi|_{x_j = x_k + 0} = \psi|_{x_j = x_k - 0}, \tag{3.3}$$

$$\left(\frac{\partial\psi}{\partial x_j} - \frac{\partial\psi}{\partial x_k}\right)\Bigg|_{x_j = x_k + 0} - \left(\frac{\partial\psi}{\partial x_j} - \frac{\partial\psi}{\partial x_k}\right)\Bigg|_{x_j = x_k - 0} = 2c\, \psi|_{x_j = x_k}. \tag{3.4}$$

Therefore, denote $T_i = (i, i+1) \in S_n$ and consider Δ_Q and Δ_{QT_i} such that $Q_i = j, Q_{i+1} = k$, then we have

$$[Q, P] + [Q, PT_i] = [QT_i, P] + [QT_i, PT_i],$$

$$i(k_{P(i)}-k_{P(i+1)})([QT_i,PT_i]-[QT_i,P]-[Q,P]+[Q,PT_i])=2c([Q,P]+[Q,PT_i]).$$

These two equations can be combined to give

$$[Q,PT_i]=\frac{c}{i(k_{P(i)}-k_{P(i+1)})-c}[Q,P]+\frac{i(k_{P(i)}-k_{P(i+1)})}{i(k_{P(i)}-k_{P(i+1)})-c}[QT_i,P]. \tag{3.5}$$

Define $\hat{T}$ to act in the space of coefficients by

$$\hat{T}[Q,P]=[QT_i,P]. \tag{3.6}$$

Then we can rewrite the above equation as

$$[Q,PT_i]=Y_i(k_{P(i)}-k_{P(i+1)})[Q,P], \quad Y_i(u)=S_R(u)+S_T(u)\hat{T}_i. \tag{3.7}$$

We can verify that

$$\begin{aligned} &Y_i(u)Y_i(-u)=I, \quad Y_i(u)Y_j(v)=Y_j(v)Y_i(u), \quad |i-j|>1,\\ &Y_i(v)Y_{i+1}(u+v)Y_i(u)=Y_{i+1}(u)Y_i(u+v)Y_{i+1}(v). \end{aligned} \tag{3.8}$$

This is the Yang-Baxter equation, which shows that all the coefficients relations are consistent, so that given $\xi_0=\xi_{id}$, we can determine all ξ_P.

In Yang's original paper, he wrote the above equation as

$$\xi_{\cdots ij\cdots}=Y_{ij}^{34}\xi_{\cdots ji\cdots}, \tag{3.9}$$

where

$$Y_{ij}^{34}=\left(y_{ij}^{-1}-1\right)+y_{ij}^{-1}P_{34}=Y_{ij}^{43}, \quad y_{ij}=1+x_{ij}, \quad x_{jk}=ic\left(p_j-p_k\right)^{-1}=-x_{kj}. \tag{3.10}$$

Each operator Y_{ij}^{ab} represents a collision of two particles in which their momenta are interexchanged.

Similarly, we can verify that

$$Y_{ij}^{ab}Y_{ji}^{ab}=1, \quad Y_{jk}^{ab}Y_{ik}^{bc}Y_{ij}^{ab}=Y_{ij}^{bc}Y_{ik}^{ab}Y_{jk}^{bc}. \tag{3.11}$$

4 Eigenvalue Problem

We have periodic boundary conditions:

$$\lambda_j\xi_0=X_{(j+1)j}X_{(j+2)j}\cdots X_{Nj}X_{1j}X_{2j}\cdots X_{(j-1)j}\xi_0, \quad j=1,2,\cdots,N, \tag{4.1}$$

where $\lambda_j=e^{ip_jL}$, $X_{ij}=T_{ij}Y_{ij}^{ij}=(1-T_{ij}x_{ij})(1+x_{ij})^{-1}$.

Also we can verify that

$$X_{ij}X_{ji}=1, \quad X_{jk}X_{ik}X_{ij}X_{kj}X_{ki}X_{ji}=1, \quad X_{ij}X_{kl}=X_{kl}X_{ij}. \tag{4.2}$$

Therefore, these N operators commute with each other.

The operators T_{ij} on ξ, which maps $N!$ elements to $N!$ elements, form a $N!\times N!$ representation of S_N. We want to reduce the representation into irreducible components.

Choose one irreducible representation R. If R = identity representation = $[N]$, then $T_{ij}=1$, and this is the bosonic case [1]. If R = antisymmetric representation = $[1^N]$, then $T_{ij} = -1$ and $X_{ij} = 1$, so that $e^{ip_jL} = 1$, which shows that there is no interaction and is consistent with the Pauli principle.

The λ_j's are functions of p, c and R. For R and $\widetilde{R}$ being conjugate representations, we have

$$\lambda_j(p;c;R) = \prod_{i\neq j}\left(\frac{1-x_{ij}}{1+x_{ij}}\right)\lambda_j(p;-c;\widetilde{R}). \tag{4.3}$$

Define $\mu_j(p;c;R)$ by

$$\mu_j\Phi = X'_{(j+1)j}X'_{(j+2)j}\cdots X'_{Nj}X'_{1j}X'_{2j}\cdots X'_{(j-1)j}\Phi, \tag{4.4}$$

where $X'_{ij} = (1+P_{ij}x_{ij})(1+x_{ij})^{-1}$. Then clearly, we have

$$\mu_j(p;c;\widetilde{R}) = \lambda_j(p;c;R). \tag{4.5}$$

Therefore, to evaluate λ_j for $R = [2^M 1^{N-2M}]$, we need to find $\mu_j(p;c;[N-M,M])$.

Consider a system of N spin-$\frac{1}{2}$ particles with M down-spins and $N-M$ up-spins, so that the spin wave functions Φ for total z spin $= \frac{1}{2}N - M$.

Consider the N spins as forming a cyclic chain. Φ has $\binom{N}{M}$ components [$N-M$ spins up , M spins down]. Then by generalized Bethe's ansatz, we have

$$\Phi = \sum_P A_P F(\Lambda_{P1},y_1)F(\Lambda_{P2},y_2)\cdots F(\Lambda_{PM},y_M), \tag{4.6}$$

where $y_1 < y_2 < \cdots < y_M$ are the "coordinates" of the "down-spins", $\Lambda_1,\Lambda_2,\cdots,\Lambda_M$ are a set of unequal numbers,

$$F(\Lambda,y) = \prod_{j=1}^{y-1}\frac{ip_j - i\Lambda - c'}{ip_{j+1} - i\Lambda + c'}, \quad c' = \frac{c}{2}. \tag{4.7}$$

$$-\prod_{j=1}^{N}\frac{ip_j - i\Lambda_\alpha - c'}{ip_j - i\Lambda_\alpha + c'} = \prod_{\beta=1}^{M}\frac{-i\Lambda_\beta + i\Lambda_\alpha + c}{-i\Lambda_\beta + i\Lambda_\alpha - c}. \tag{4.8}$$

$$\mu_j(p;c;[N-M,M]) = \prod_{\beta=1}^{M}\frac{ip_j - i\Lambda_\beta - c'}{ip_j - i\Lambda_\beta + c'} \tag{4.9}$$

Therefore, we need to solve

$$e^{ip_jL} = \prod_{\beta=1}^{M}\frac{ip_j - i\Lambda_\beta - c'}{ip_j - i\Lambda_\beta + c'}, \quad j = 1,2,\cdots,N, \tag{4.10}$$

Define

$$\theta(p) = -2\arctan\left(\frac{p}{c}\right), \quad -\pi \leqslant \theta < \pi. \tag{4.11}$$

Notice that

$$\begin{aligned} e^{i\theta(p)} &= -\sin\left(2\arctan\left(\frac{p}{c}\right)\right) + i\cos\left(2\arctan\left(\frac{p}{c}\right)\right) \\ &= -\frac{2pc}{c^2+p^2} + i\frac{c^2-p^2}{c^2+p^2} = \frac{1-i\frac{p}{c}}{1+i\frac{p}{c}}. \end{aligned} \tag{4.12}$$

Hence for the $R_\psi = [2^M 1^{N-2M}]$ symmetry and N even, M odd case, taking logarithm on both sides of equations (4.8) and (4.10), we have

$$-\sum_{i=1}^{N}\theta(2\Lambda - 2p_i) = 2\pi J_\Lambda - \sum_{\beta=1}^{M}\theta(\Lambda-\Lambda_\beta), \quad Lp = 2\pi I_p + \sum_{\Lambda}\theta(2p-2\Lambda), \tag{4.13}$$

where J_Λ = successive integers from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$, and $\frac{1}{2}+I_p$ = successive integers from $1-\frac{N}{2}$ to $\frac{N}{2}$.

$$-\int_{-Q}^{Q}\theta(2\Lambda-2p)\rho(p)\mathrm{d}p = 2\pi g - \int_{-B}^{B}\theta(\Lambda-\Lambda')\sigma(\Lambda')\mathrm{d}\Lambda', \tag{4.14}$$

$$p = 2\pi f + \int_{-B}^{B}\theta(2p-2\Lambda)\sigma(\Lambda)\mathrm{d}\Lambda, \tag{4.15}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\Lambda} = \sigma, \quad \frac{\mathrm{d}f}{\mathrm{d}p} = \rho. \tag{4.16}$$

After differentiation, we have

$$2\pi\sigma = -\int_{-B}^{B}\frac{2c\sigma(\Lambda')\mathrm{d}\Lambda'}{c^2+(\Lambda-\Lambda')^2} + \int_{-Q}^{Q}\frac{4c\rho(p)\mathrm{d}p}{c^2+4(p-\Lambda)^2}, \tag{4.17}$$

$$2\pi\rho = 1 + \int_{-B}^{B}\frac{4c\sigma(\Lambda)\mathrm{d}\Lambda}{c^2+4(\Lambda-p)^2}. \tag{4.18}$$

$$\frac{N}{L} = \int_{-Q}^{Q}\rho(p)\mathrm{d}p, \quad \frac{M}{L} = \int_{-B}^{B}\sigma(\Lambda)\mathrm{d}\Lambda, \quad \frac{E}{L} = \int_{-Q}^{Q}p^2\rho(p)\mathrm{d}p. \tag{4.19}$$

References

[1] Elliott H. Lieb and Werner Liniger. Exact analysis of an interacting bose gas. i. the general solution and the ground state. *Phys. Rev.*, 130:1605–1616, May 1963.

[2] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, Dec 1967.